Stat 511 Exam1

February 26, 2008
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I have neither given nor received unauthorized assistance on this exam.

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Name

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Name Printed

1. Consider a problem of quadratic regression in one variable, x. In particular, suppose that $n=5$ values of a response y are related to values $x=0,1,2,3,4$ by a linear model $\mathbf{Y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$ for

$$\mathbf{Y}=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix},\ \mathbf{X}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix},\ \boldsymbol{\beta}=\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix},\ \text{and}\ \boldsymbol{\varepsilon}=\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}$$

a) A cofactor expansion of the determinant of a 3×3 matrix $\mathbf{A}=(a_{ij})$ on its first row is

$$\det(\mathbf{A})=a_{11}\det(\mathbf{M}_{11})-a_{12}\det(\mathbf{M}_{12})+a_{13}\det(\mathbf{M}_{13})$$

where $\mathbf{M}_{ij}$ is obtained from $\mathbf{A}$ by deleting its ith row and jth column. *Use this fact* and argue that $\mathbf{X}$ is of full rank.

b) Define

$$\mathbf{F}=\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\ \text{so that } \mathbf{F} \text{ is nonsingular and } \mathbf{F}^{-1}=\begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\mathbf{W}=\mathbf{XF}$. Argue carefully that $C(\mathbf{W})=C(\mathbf{X})$.

c) Notice that $\mathbf{W}'\mathbf{W}$ is diagonal. Suppose that $\mathbf{Y}' = (-2, 0, 4, 2, 2)$. Find the OLS estimate of $\boldsymbol{\gamma}$ in the model $\mathbf{Y} = \mathbf{W}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}$ and then OLS estimate of $\boldsymbol{\beta}$ in the original model. (Find numerical values.)

$\hat{\boldsymbol{\gamma}}_{\text{OLS}} =$

$\hat{\boldsymbol{\beta}}_{\text{OLS}} =$

d) As it turns out, for the set of observations in c), $SSE = 128/35$. What then is an estimated variance-covariance matrix for $\hat{\boldsymbol{\gamma}}_{\text{OLS}}$? What is the corresponding estimated variance-covariance matrix for $\hat{\boldsymbol{\beta}}_{\text{OLS}}$? (No need to simplify/do matrix multiplications.)

e) Based on your answer to d) give 95% confidence limits for the difference in mean responses for $x=3$ and $x=1$ in the normal Gauss-Markov model. (No need to simplify. If you don't have appropriate tables with you, tell me exactly what quantile(s) of what distribution you need.)

f) Consider the hypothesis

$$\mathrm{H}_0 : \left[x=0 \text{ and } x=4 \text{ have the same mean response} \right] \text{ and } \left[x=1 \text{ and } x=3 \text{ have the same mean response} \right]$$

(Notice that in this model this is the hypothesis that the quadratic regression function has a critical point at $x=2$.) Show how to test this hypothesis. (Show how to compute an appropriate test statistic and say exactly what null distribution you could use.) Be careful. Some ways of writing this hypothesis may not produce a "testable" hypothesis.

2. Notice that the first characterization of the estimability of $\mathbf{c}'\boldsymbol{\beta}$ in the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ can be rephrased as "$\mathbf{c}'\boldsymbol{\beta}$ is estimable iff there is a linear combination of the variables $y_1, y_2, \ldots, y_n$ with expected value $\mathbf{c}'\boldsymbol{\beta}$ for all $\boldsymbol{\beta}$."

a) Consider the "effects model version of the 3-group/2-observations-per-group" one-way ANOVA model used repeatedly as an example in lecture early in the term (version 2) of example b)). Using the characterization of estimability above, show that $\mu + \frac{1}{3}(\tau_1 + \tau_2 + \tau_3)$ is estimable.

b) In the context of part a) above, give formulas for *both* the BLUE of $\mu + \frac{1}{3}(\tau_1 + \tau_2 + \tau_3)$ AND an unbiased linear estimator of this quantity that is NOT the BLUE.

3. Attached to this exam is an `R` printout made in the analysis of data from a pilot plant chemical process. Process yield is thought to depend upon condensation temperature and amount of boron employed. A "quadratic response surface" model of the form

$$y = \beta_0 + \beta_1 t + \beta_2 B + \beta_3 t^2 + \beta_4 B^2 + \beta_5 tB + \varepsilon$$

has been fit to the yield data (and by the way, predicts a maximum yield near $t = -1$ and $B = .5$).

a) Give the value of an F statistic and degrees of freedom for testing the hypothesis that $\mathrm{E}\mathbf{Y} \in C((\mathbf{1} \,|\, \mathbf{t} \,|\, \mathbf{B}))$ in this model.

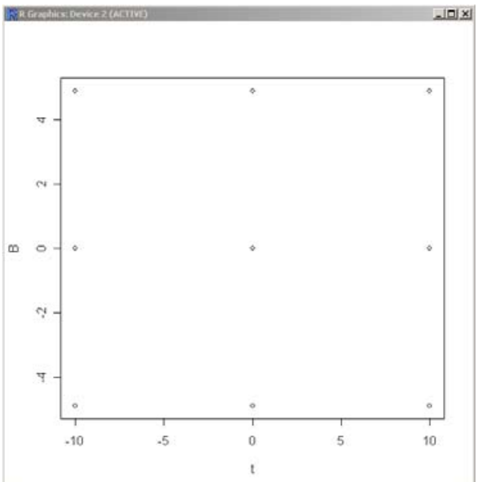
$F =$ __________ d.f. $=$ _____ , _____

b) Which observation (or observations) appears (appear) to be most influential in determining the character of the fitted surface? "Where" is (are) that (those) observation (observations) in (t, B)-space? (Note that there is a plot on the printout.) Provide rationale for your choice.

```
> chemical
  yield time temp Bamt
1  21.1  150   90 24.4
2  23.7   10   90 29.3
3  20.7    8   90 34.2
4  21.1   35  100 24.4
5  24.1    8  100 29.3
6  22.2    7  100 34.2
7  18.4   18  110 24.4
8  23.4    8  110 29.3
9  21.9   10  110 34.2

> t<-temp-100
> B<-Bamt-29.3

> plot(t,B)
```




```
> t2<-t^2
> B2<-B^2
> tB<-t*B

> quadyield<-lm(yield~t+B+t2+B2+tB)

> summary(quadyield)

Call:
lm(formula = yield ~ t + B + t2 + B2 + tB)

Residuals:
       1        2        3        4        5        6        7        8        9
-0.06389 -0.02222  0.08611  0.27778 -0.25556 -0.02222 -0.21389  0.27778 -0.06389

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 24.355556   0.228499 106.589 1.82e-06 ***
t           -0.030000   0.012515  -2.397 0.096130 .
B            0.142857   0.025542   5.593 0.011289 *
t2          -0.009333   0.002168  -4.306 0.023061 *
B2          -0.118006   0.009028 -13.070 0.000967 ***
tB           0.019898   0.003128   6.361 0.007863 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3066 on 3 degrees of freedom
Multiple R-squared: 0.9889,     Adjusted R-squared: 0.9704
F-statistic: 53.37 on 5 and 3 DF,  p-value: 0.00394


> vcov(quadyield)
              (Intercept)             t             B            t2            B2            tB
(Intercept)  5.221193e-02  1.688218e-19  5.681659e-20 -3.132716e-04 -1.304755e-03 -6.622578e-20
t            1.688218e-19  1.566358e-04  5.186088e-21 -7.078389e-22 -9.409579e-21 -4.449178e-22
B            5.681659e-20  5.186088e-21  6.523774e-04 -6.614999e-22 -4.488606e-21  2.036626e-21
t2          -3.132716e-04 -7.078389e-22 -6.614999e-22  4.699074e-06  1.507303e-22  7.738612e-22
B2          -1.304755e-03 -9.409579e-21 -4.488606e-21  1.507303e-22  8.151321e-05  1.951499e-21
tB          -6.622578e-20 -4.449178e-22  2.036626e-21  7.738612e-22  1.951499e-21  9.785660e-06
```

```
> anova(quadyield)
Analysis of Variance Table

Response: yield
          Df  Sum Sq Mean Sq  F value    Pr(>F)
t          1  0.5400  0.5400   5.7458 0.0961299 .
B          1  2.9400  2.9400  31.2828 0.0112892 *
t2         1  1.7422  1.7422  18.5379 0.0230607 *
B2         1 16.0556 16.0556 170.8374 0.0009672 ***
tB         1  3.8025  3.8025  40.4601 0.0078628 **
Residuals  3  0.2819  0.0940
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1


> predict(quadyield,se.fit=T)
$fit
       1        2        3        4        5        6        7        8        9
21.16389 23.72222 20.61389 20.82222 24.35556 22.22222 18.61389 23.12222 21.96389

$se.fit
[1] 0.2751496 0.2284993 0.2751496 0.2284993 0.2284993 0.2284993 0.2751496 0.2284993 0.2751496

$df
[1] 3

$residual.scale
[1] 0.306564

> ones<-rep(1,9)
> X<-cbind(ones,t,B,t2,B2,tB)

> X%*%ginv(t(X)%*%X,tol=1e-15)%*%t(X)
             [,1]       [,2]        [,3]       [,4]       [,5]       [,6]        [,7]       [,8]
 [1,]  0.80555556  0.2222222 -0.02777778  0.2222222 -0.1111111 -0.1111111 -0.02777778 -0.1111111
 [2,]  0.22222222  0.5555556  0.22222222 -0.1111111  0.2222222 -0.1111111 -0.11111111  0.2222222
 [3,] -0.02777778  0.2222222  0.80555556 -0.1111111 -0.1111111  0.2222222  0.13888889 -0.1111111
 [4,]  0.22222222 -0.1111111 -0.11111111  0.5555556  0.2222222  0.2222222  0.22222222 -0.1111111
 [5,] -0.11111111  0.2222222 -0.11111111  0.2222222  0.5555556  0.2222222 -0.11111111  0.2222222
 [6,] -0.11111111 -0.1111111  0.22222222  0.2222222  0.2222222  0.5555556 -0.11111111 -0.1111111
 [7,] -0.02777778 -0.1111111  0.13888889  0.2222222 -0.1111111 -0.1111111  0.80555556  0.2222222
 [8,] -0.11111111  0.2222222 -0.11111111 -0.1111111  0.2222222 -0.1111111  0.22222222  0.5555556
 [9,]  0.13888889 -0.1111111 -0.02777778 -0.1111111 -0.1111111  0.2222222 -0.02777778  0.2222222
             [,9]
 [1,]  0.13888889
 [2,] -0.11111111
 [3,] -0.02777778
 [4,] -0.11111111
 [5,] -0.11111111
 [6,]  0.22222222
 [7,] -0.02777778
 [8,]  0.22222222
 [9,]  0.80555556
```