KEY

Stat 511 Final Exam

May 8, 2003 Prof. Vardeman

1. A classic data set of Wishart (later analyzed by Ostle and then Stapleton) concerns weight gains of young pigs in a comparative feeding trial. 3 different foods were fed to pigs of both sexes. 5 male pigs and 5 female pigs were assigned to each food (making a total of 30 pigs in the study). (There were also 5 pens involved in this study, but that is a detail we shall initially ignore.) With

y_{ijk} = the weight gain of the kth pig of sex j fed food i

x_{ijk} = the initial weight of the kth pig of sex j fed food i

first consider the model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \theta x_{ijk} + \varepsilon_{ijk} \qquad (*)$$

where constants $\alpha_i, \beta_j, \alpha\beta_{ij}$ for $i = 1,2,3$ and $j = 1,2$ are subject to the sum restrictions, μ and θ are constants, and the ε_{ijk} are iid $\mathrm{N}(0,\sigma^2)$. (On **Printout #1** Sex is coded "1" for Male and "2" for Female.)

10 pts a) Under model (*) give 95% confidence limits for the standard deviation of weight gain for a pig of fixed sex and weight fed a fixed food. (Plug in numbers, but don't do arithmetic.)

Use $\left(\sqrt{\frac{SSE}{\chi^2_U}}, \sqrt{\frac{SSE}{\chi^2_L}}\right)$ i.e. $\left(\sqrt{\frac{7.1745}{38.076}}, \sqrt{\frac{7.1745}{11.689}}\right)$

$(.43, .78)$

9 pts b) In this model (that contains both continuous and categorical predictor variables for the balanced data set) what is the expected value of $\bar{y}_{1..} - \bar{y}_{2..}$? Is it $\alpha_1 - \alpha_2$?

$\bar{y}_{i..} = \mu + \alpha_i + \theta\bar{x}_{i..} + \bar{\varepsilon}_{i..}$ So

$\bar{y}_{1..} - \bar{y}_{2..} = \alpha_1 - \alpha_2 + \theta(\bar{x}_{1..} - \bar{x}_{2..}) + \bar{\varepsilon}_{1..} - \bar{\varepsilon}_{2..}$

And $E(\bar{y}_{1..} - \bar{y}_{2..}) = \alpha_1 - \alpha_2 + \theta(\bar{x}_{1..} - \bar{x}_{2..})$

which is, in general, <u>not</u> $\alpha_1 - \alpha_2$

9 pts c) Interpret (in words) $\beta_1 - \beta_2$. This quantity is estimable. Name, for example, a linear combination of $\bar{y}_{.1.}, \bar{y}_{.2.}, y_{111}$, and y_{112} that has this expected value.

This is, <u>for a fixed initial weight</u>, the simple average across the 3 foods of a male weight gain minus the same for a female.

$E(\bar{y}_{.1.} - \bar{y}_{.2.}) = \beta_1 - \beta_2 + \theta(\bar{x}_{.1.} - \bar{x}_{.2.})$ while $E(y_{111} - y_{112}) = 3\theta$

So $E\left(\bar{y}_{.1.} - \bar{y}_{.2.} - \frac{1}{3}(y_{111} - y_{112})(\bar{x}_{.1.} - \bar{x}_{.2.})\right)$
$= \beta_1 - \beta_2$

8pts d) Make and interpret 95% confidence limits for θ.

Use $\hat{\theta} \pm t\ s.e.\hat{\theta}$ i.e. $.07558 \pm 2.069(.01725)$

$.076 \pm .036$

These are limits on the additional mean weight gain that accompanies a 1 lb increase in initial weight for either sex and any food.

13pts e) Show how to compute 95% confidence limits for $\alpha_1 - \alpha_2$. (Plug in, but you don't need to do arithmetic.)

Use $a_1 - a_2 \pm t\ s.e.(a_1 - a_2)$ This is

$$(.36734 - (-.06168)) \pm 2.069(.5585)\sqrt{(1,-1)\begin{pmatrix} .0667 & -.0335 \\ -.0335 & .0670 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

13pts f) Find ~~90~~ 95% prediction limits for the weight gain of an additional 38 lb male pig fed feed #1. (Plug in but you don't need to do arithmetic.)

This is a pig like the 1st one listed – For this pig, we have

$t\ s.e.\hat{y} = 9.906914 - 9.379249 = .5277$

And $t\sqrt{MSE} = 2.069(.5585) = 1.1555$

So prediction limits are

$$9.3792 \pm \sqrt{(1.1555)^2 + (.5277)^2}$$

As it turns out, the pigs in this study were raised in 5 pens (each of which had one pig of each Food $\times$ Sex combination in it). With $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ iid $\mathrm{N}(0, \sigma_\gamma^2)$ independent of the ε_{ijk} and

$l(ijk)$ = the pen number (l, from 1 to 5) for pig ijk

consider the mixed model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \theta x_{ijk} + \gamma_{l(ijk)} + \varepsilon_{ijk} \qquad (**)$$

and an analysis of weight gain data based on it. (Pigs in the same pen share a random effect in (**).)

g) In the mixed effects model (**), weight gains for pigs in the same pen are correlated. What is this correlation (in terms of the model parameters)?

7pts

This is $\dfrac{\sigma_\gamma^2}{\sigma_\gamma^2+\sigma^2}$

h) Give approximate 95% confidence limits for the standard deviations σ and σ_γ in (**).

7pts

For σ: (.365, .685)

For σ_γ: (.078, .841)

9pts

i) The quantity $\mu+\alpha_1+\beta_1+\alpha\beta_{11}+38\theta$ is what the model (**) gives as the mean (across all possible pens) weight gain for a 38 lb male pig fed food 1. (Note that the first pig in the data set was a male of this size fed food 1.) Give an estimate of this quantity and show how to compute a standard error. Notice that there is an estimated variance-covariance matrix for the estimates of fixed effects printed on the output. You don't have to copy it onto this page. Just call it what it is called on the printout, and write an (otherwise numerical) expression involving it.

Estimate: 9.363281

$(=\hat{\mu}+a_1+b_1+ab_{11}+38\hat{\theta})$

Standard Error: Here we want

$$\sqrt{(1,1,0,1,38,1,0)\ W \begin{pmatrix}1\\1\\0\\1\\38\\1\\0\end{pmatrix}} = \sqrt{.06659} = .258$$

9pts

j) If γ^* is a new random effect for a new pen and ε^* is a new random error associated with a new male pig that happens to weigh 38 lbs and is to be fed food 1, and these are independent of each other and all of the ε_{ijk} and η_l for the data in hand, one might wish to predict $\mu+\alpha_1+\beta_1+\alpha\beta_{11}+38\theta+\gamma^*+\varepsilon^*$. Give a sensible standard error of prediction for this. (If it's relevant, you may abbreviate your answer to part i) as "SE".)

Use $\sqrt{\hat{\sigma}^2+\hat{\sigma}_\gamma^2+(SE)^2} = \sqrt{(.5004)^2+(.2569)^2+(SE)^2}$

2. A Ni-Cad battery manufacturer was interested in finding what set of process conditions produces the smallest fraction of cells with "shorts." For $2 \times 2 \times 2 = 8$ different process set-ups, counts were made of batteries with shorts. The sample sizes varied set-up to set-up (from 88 to 100). Factors and their levels in the study were

A- Nylon Sleeve	1-no vs 2-yes
B- Rolling Direction	1-lower edge first vs 2-upper edge first
C- Rolling Order	1-negative first vs 2-positive first

Printout #2 summarizes two (binomial) GLM analyses of the manufacturer's data. (The data matrix M has counts of shorts in column 1 and counts of nonshorts in column 2.) The first of the two analyses was done using a logit link and the second used a probit link.

8pts a) Does it appear from these data and analyses that any of these 3 factors influence the rate of short production? Is it plausible that the differences between observed rates is "just noise"? Explain.

In both analyses the effects of all of A, B, C are "statistically significant." It is not plausible that apparent differences among set-ups are just noise.

8pts b) Notice that there were 4 of 8 different process set-ups which produced no observed shorts. Which of these would you recommend for future running of the production process? Why?

For both analyses $a_1 > 0$, $b_1 > 0$ and $c_1 < 0$ (which by the sum restriction makes $a_2 < 0$, $b_2 < 0$ and $c_2 > 0$). So for smallest $\underset{\sim}{x}'\underset{\sim}{b}$ and smallest $\hat{p}$ we want level 2 of A, level 2 of B and level 1 of C. Note that this combination has the smallest $\hat{p}$ for both models. (These are the conditions for the 7th set-up.)

9pts c) Give approximate 95% confidence limits for the rate of shorts associated with your choice from part b). Do the estimation first based on the logit, then based on the probit link. How much difference is there between the two set of limits?

logit: $.0001345 \pm 1.96(.0002086)$

probit: $.0000176 \pm 1.96(.0000467)$

These are substantially different (though both give very small estimates of p). Logit and probit analyses are typically quite similar for, say, $.1 < p < .9$. They can differ substantially at the extremes. Here, with $n \approx 100$ there is no way to distinguish which model is best.

3. Two different processes (laser drilling and electrical discharge machining) were used to make holes in miniature high precision metal parts. These were supposed to be at a 45° angle with the surface of the part. Measured angles for 13 parts drilled by each process are analyzed on **Printout #3**.

7pts a) Under the assumption that these are two independent samples from normal distributions of angles, standard facts about sample variances imply that

$$\left(\frac{12s^2_{\text{laser}}}{\sigma^2_{\text{laser}}}\right) \text{ and } \left(\frac{12s^2_{\text{edm}}}{\sigma^2_{\text{edm}}}\right)$$

are independent χ^2_{12} random variables. Use this fact and percentage points of the F distribution to make 90% confidence limits for $\sigma_{\text{laser}}/\sigma_{\text{edm}}$ (which could be used to compare precisions of the two drilling methods). (Plug in, but you need not simplify.)

$$R = \frac{s^2_{laser}/\sigma^2_{laser}}{s^2_{edm}/\sigma^2_{edm}} \sim F_{12,12} \quad \text{So } P[F_L < R < F_u] = \text{confidence level}$$

$$\Rightarrow \left(\frac{s^2_{laser}}{s^2_{edm}F_u}, \frac{s^2_{laser}}{s^2_{edm}F_L}\right) \text{ is a confidence interval for } \frac{\sigma^2_{laser}}{\sigma^2_{edm}}$$

$$\text{So use } \left(\frac{s_{laser}}{s_{edm}\sqrt{F_u}}, \frac{s_{laser}}{s_{edm}}\sqrt{F_u}\right) \text{ i.e. } \left(\frac{2.246}{.817\sqrt{2.69}}, \frac{2.246}{.817}\sqrt{2.69}\right)$$

Examination of plots for the two samples suggests that normal distribution model assumptions may not be appropriate. So, consider instead an analysis of based on bootstrap ideas.

6pts b) What is the value of a "bootstrap standard error" for the statistic $s_{\text{laser}}/s_{\text{edm}}$?

1.117

6pts c) What value would you subtract from the observed ratio $s_{\text{laser}}/s_{\text{edm}}$ in order to try and correct for bias in estimating $\sigma_{\text{laser}}/\sigma_{\text{edm}}$?

Estimated bias is $2.746609 - \frac{2.245937}{.8169675} = -.0025$

(So a biased-corrected estimate would be $\frac{2.245937}{.8169675} - (-.0025)$)

6pts d) What are (uncorrected) 90% percentile bootstrap confidence limits for $\sigma_{\text{laser}}/\sigma_{\text{edm}}$? Explain.

Reading the lower and upper .05 points of the bootstrapped ratio dsn, we want roughly

(1.2, 4.5)

(This interval has about the same upper end as the normal thy interval, but has a substantially smaller lower end.)

4. Fitting sinusoids to noisy data can be done via nonlinear least squares. **Printout #4** concerns doing just this. 41 data pairs (x_i, y_i) have a sinusoidal pattern and R's `nle` has been used to fit the model

$$y_i = A\sin(k(x_i - x_0)) + \varepsilon_i$$

to them by estimation of parameters A, k, x_0 (and σ).

4pts a) A different set of starting values not shown on the printout led to a fit with $\hat{A} = -2.76987, \hat{k} = 6.39168$, and $\hat{x}_0 = 0.04980$ and $SSE = 20.94946$. This final error sum of squares is exactly as for the fit on Printout #4, while the estimates for A and x_0 are quite different. This shows that the function that `nle` optimizes has multiple local optima. Why should this have been obvious before beginning in this situation?

The periodic nature of the sin function means that there are lots of different sets of parameters (involving a shift by whole periods or by odd multiples of a half period with a change in sign for A) that produce the same curve and therefore same fits and SSE.

(In light of the above point, in order to talk rationally about inference for parameters, we would really need to do something like require that $A > 0$ and $k > 0$ and put some restriction on x_0. We'll assume that has been done and that the algorithm has converged to a global optimum subject to these.)

8pts b) What are approximate 95% confidence limits for $A\sin(k(1-x_0))$, the mean response at $x = 1$? (Plug in, but don't try to simplify.) Use $f(\underline{x}, \underline{b}_{OLS}) \pm t\sqrt{\hat{G}\, MSE(\hat{D}'\hat{D})^{-1}\hat{G}'}$

Note that $\frac{\partial}{\partial A} A\sin(k(x-x_0)) = \sin(k(x-x_0))$

$\frac{\partial}{\partial k} A\sin(k(x-x_0)) = A(x-x_0)\cos(k(x-x_0))$ and $\frac{\partial}{\partial x_0} A\sin(k(x-x_0)) = -Ak\cos(k(x-x_0))$

Plug in $f(\underline{x}, \underline{b}_{OLS}) = 2.7699\sin(6.39168(1-.54131))$ $\quad t = 2.0244$

$\hat{G} = (\sin(6.39168(1-.54131)), 2.7699(1-.54131)\cos(6.39168(1-.54131)), -2.7699(6.39168)\cos(6.39168(1-.54131)))$

and $MSE(\hat{D}'\hat{D})^{-1} = \text{vcov(nlsfit.1)}$

4pts c) Give approximate 95% confidence limits for the period of the sinusoid, $\frac{k}{2\pi}$.

Use confidence limits for k divided by 2π – i.e.

$$\frac{6.39168}{2\pi} \pm 2.0244\,\frac{.09408}{2\pi}$$

or directly from the printout (using the profile likelihood idea)

$$\left(\frac{6.229}{2\pi}, \frac{6.549}{2\pi}\right)$$