Stat 511 Exam 2

April 11, 2003 Prof. Vardeman

1. The first printout attached to this exam concerns the analysis of some data from a tire grip study. Drag, x (measured in %), is applied to a tire and the (grip) force, y (measured in lbs), with which it holds the pavement is measured. Physical theory suggests that

$$y \approx \alpha \exp(\beta x)$$

We'll consider two analyses of $n = 19$ data pairs (x_i, y_i). The first is one based on the model

$$\ln y_i = \ln \alpha + \beta x_i + \varepsilon_i \qquad (*)$$

and the second is based on the model

$$y_i = \alpha \exp(\beta x_i) + \varepsilon_i \qquad (**)$$

(where in both cases the errors ε_i are iid $\mathrm{N}(0, \sigma^2)$). (Actually, simple plots of $\ln y$ vs x and of y vs x call into question the constant variance assumptions. But we will ignore this concern for the narrow purposes of this exam.)

7pts a) Based on the linear model in (*), give 95% confidence limits for α (not $\ln \alpha$) and 95% limits for β. (Plug in, but you need not do arithmetic.)

limits for $\ln\alpha$ are $6.40 \pm 2.110(.0517)$ so limits for α are

$$\left(e^{6.40-2.110(.0517)}, e^{6.40+2.110(.0517)}\right)$$

limits for β are $-.0102 \pm 2.110(.000875)$

7pts b) Give approximate 95% confidence limits for α and β based on the nonlinear model (**).

for α: $601.58 \pm 2.110(28.603)$ or using profile loglikelihood $(543.4, 663.5)$

for β: $-.0101 \pm 2.110(.001161)$ or using profile loglikelihood $(-.0126, -.0077)$

7pts c) Give (any set of) approximate 90% confidence limits for σ in the nonlinear model (**). (Plug in numbers, but you need not do arithmetic.)

Use the approximation $\frac{SSE}{\sigma^2} \sim \chi^2_{17}$ – Then limits are

$$\left(\sqrt{\frac{17(MSE)}{\chi^2_U}}, \sqrt{\frac{17(MSE)}{\chi^2_L}}\right)$$

i.e. $\left(51.12\sqrt{\frac{17}{27.587}}, 51.12\sqrt{\frac{17}{8.672}}\right)$

my intention was to write $x = 50$ (not $x = .5$)

10pts d) Find approximate 95% prediction limits for the next grip force if $x = .5$, based on the nonlinear model (**). (There is enough information on the printout to allow you to apply a bit of calculus and some simple matrix calculations to get this. When you have reduced your answer to a completely numerical expression, you may stop without doing arithmetic.)

We want to use $f(\underset{\sim}{x}, \underset{\sim}{b}_{OLS}) \pm \underbrace{t\sqrt{MSE}\sqrt{1+\hat{G}(\hat{D}'\hat{D})^{-1}\hat{G}'}}_{t\sqrt{MSE+\hat{G}(MSE(\hat{D}'\hat{D})^{-1})\hat{G}'}}$

$f(x, b_{OLS}) = 601.98\, e^{-.01009(.5)}$

$t = 2.110 \quad MSE = \frac{44419.51}{17}$

$$MSE(\hat{D}'\hat{D})^{-1} = \begin{pmatrix} (28.6)^2 & (28.6)(.00116)(-.7887) \\ & (.00116)^2 \end{pmatrix}$$

and since $\frac{\partial f}{\partial \alpha} = e^{x\beta}$ and $\frac{\partial f}{\partial \beta} = \alpha x e^{x\beta}$

$$\hat{G} = \left(e^{.5(-.01009)},\ 601.98\,(.5)\,e^{.5(-.01009)}\right)$$

7pts e) The graph on the printout is a contour plot for the error sum of squares. Based on this plot, how plausible is the hypothesis $H_0: \alpha = 650$ and $\beta = -.007$? Make some quantitative statement if you can. (It's possible, for example, to get a rough idea of a p-value from the plot.)

This hypothesis is implausible. "SSE" for this pair of parameters would exceed 80,000. 80,000/44,420 = 1.8010. Setting

$$1.8010 = \left(1 + \frac{2}{17}F\right)$$

we get $F = 6.808$ – This exceeds the upper .01 pt of $F_{2,17}$ – Thus p-value < .01 (a 99% confidence region fails to contain $\alpha = 650$, $\beta = -.007$)

6pts f) A model is linear in its (mean) parameters exactly when the error sum of squares is quadratic in those parameters (and ~~these~~ produces exactly elliptical level contours). In light of this fact, what does the substantial agreement between answers to parts a) and b) and the appearance of the contour plot indicate about model (**) and the inferences based on it?

The contours are nearly elliptical (and inferences based on taking logarithms and using a linear model don't differ much from the approximate ones based the non linear model) and this suggests that 1) the non-linear model is not terribly non linear and 2) the corresponding inferences may be nearly "exact."

2. Consider the small mixed model

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \underset{6\times 1}{\mathbf{1}}\mu + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \underset{6\times 1}{\boldsymbol{\varepsilon}}$$

for $\mathbf{G} = \sigma_u^2 \underset{4\times 4}{\mathbf{I}}$ and $\mathbf{R} = \sigma^2 \underset{6\times 6}{\mathbf{I}}$.

7pts a) What is the variance-covariance matrix for $\mathbf{Y}$ in this model?

$$V = \left(\begin{array}{cc|cc|c|c} \sigma^2+\sigma_u^2 & \sigma_u^2 & & & & \\ \sigma_u^2 & \sigma^2+\sigma_u^2 & & & & \\ \hline & & \sigma^2+\sigma_u^2 & \sigma_u^2 & & \\ & & \sigma_u^2 & \sigma^2+\sigma_u^2 & & \\ \hline & & & & \sigma^2+\sigma_u^2 & \\ \hline & & & & & \sigma^2+\sigma_u^2 \end{array}\right)$$

7pts b) REML estimation of variance components in this model is maximum likelihood estimation of variance components based on $\underset{5\times 6}{\mathbf{B}}\mathbf{Y}$ for an appropriate $\mathbf{B}$. Give one such $\mathbf{B}$.

We need a matrix of rank 5 that has rows that are $\perp$ to a vector of 1's. One possibility is

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The second printout attached to this exam enables a mixed effects analysis for the observation vector $\mathbf{Y} = (.7, .6, 2.1, 2.3, .1, 1.5)'$.

7pts c) The numerical maximization of the restricted likelihood produces $\hat{\sigma} = .1118$. There is another route to producing essentially this value. Show an elementary calculation that produces this value.

$s_1^2 = \frac{1}{2-1}\left((.7-.65)^2 + (.6-.65)^2\right)$ estimates σ^2 as does

$s_2^2 = \frac{1}{2-1}\left((2.1-2.2)^2 + (2.3-2.2)^2\right)$. We could pool these to get

$$s^2_{pooled} = \frac{(.7-.65)^2 + (.6-.65)^2 + (2.1-2.2)^2 + (2.3-2.2)^2}{1+1}$$

$$= \frac{.0250}{2} = .0125$$

and $\sqrt{.0125} = .1118$

7pts d) Give 95% exact confidence limits for σ. (Show some work. Plug in numerical values, but you don't need to do arithmetic.)

s^2_{pooled} as in e) is such that $\frac{2 s^2_{pooled}}{\sigma^2} \sim \chi^2_2$ —
This leads to confidence limits for σ of

$$\left(s_{pooled}\sqrt{\frac{2}{\chi^2_U}} \; , \; s_{pooled}\sqrt{\frac{2}{\chi^2_L}} \right) \text{ i.e. } \left(.1118\sqrt{\frac{2}{7.387}} \; , \; .1118\sqrt{\frac{2}{.051}} \right)$$

7pts e) What are 95% approximate confidence limits for σ_u?

(.4117, 2.0608) directly from the printout

7pts f) Observations y_1 and y_2 are from "group 1." How do you suggest predicting another (unobserved) value from this group? (Give a numerical value and say what it is in terms used in lecture.) Would you use the same numerical value for predicting if instead of the mixed model we used a fixed effects model with 4 unknown group means and $\sigma^2\mathbf{I}$ for a variance-covariance matrix? (If not, what value would you use?)

predict with $1.1136 + (-.4602) = .6534$

This is the approximate BLUP of $\ell = \mu + u_1$,
namely $\underset{\substack{\| \\ 1.114}}{\hat{\mu}} + \hat{u}_1$

Under the fixed effects model I'd use $\frac{1}{2}(y_1+y_2) = .65$ which is slightly smaller than this. (The random effects analysis shrinks the prediction in the direction of $\hat{\mu}$ for all 4 groups.)

3. In the study of the precision of a measuring device, each of $a = 2$ widgets (call widgets levels of Factor A) was measured $m = 2$ times by each of $b = 5$ different technicians (call "technicians" levels of Factor B). The resulting data can be thought of as having 2×5 (complete, balanced, replicated) factorial structure. With

$$y_{ijk} = \text{measurement } k \text{ by technician } j \text{ on widget } i$$

model as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk}$$

where μ is the only fixed effect, all the random effects are independent, the $\alpha_i \sim \mathrm{N}(0, \sigma_\alpha^2)$, the $\beta_j \sim \mathrm{N}(0, \sigma_\beta^2)$, the $\alpha\beta_{ij} \sim \mathrm{N}(0, \sigma_{\alpha\beta}^2)$, and the $\varepsilon_{ijk} \sim \mathrm{N}(0, \sigma^2)$. Standard 2-way factorial ANOVA calculations were done and produced the following ANOVA table.

Source	***df***	***MS***	**EMS**
A	1	20.0	$\sigma^2 + 2\sigma_{\alpha\beta}^2 + 10\sigma_\alpha^2$
B	4	10.0	$\sigma^2 + 2\sigma_{\alpha\beta}^2 + 4\sigma_\beta^2$
AB	4	6.0	$\sigma^2 + 2\sigma_{\alpha\beta}^2$
Error	10	2.0	σ^2

7pts a) A quantity of serious interest in this context is $\sigma_\beta^2 + \sigma_{\alpha\beta}^2$ (which is called a measure of measurement "reproducibility"). Find a sensible point estimate of this quantity.

$\widehat{\sigma_{\alpha\beta}^2} = (MSAB - MSE)/2$ and $\widehat{\sigma_\beta^2} = (MSB - MSAB)/4$

Summing these two I get

$$\widehat{\sigma_\beta^2 + \sigma_{\alpha\beta}^2} = \frac{1}{4}MSB + \frac{1}{4}MSAB - \frac{1}{2}MSE$$
$$= \frac{1}{4}(10) + \frac{1}{4}(6) - \frac{1}{2}(2) = 3$$

7pts b) Make approximate 90% confidence limits for $\sigma_\beta^2 + \sigma_{\alpha\beta}^2$. (As usual, you should plug in numbers, but you don't need to do arithmetic.)

Use Satterthwaite $\nu = \dfrac{3^2}{\frac{(2.5)^2}{4} + \frac{(1.5)^2}{4} + \frac{(1)^2}{10}} = 4.04$

round down and use limits

$$\left(\frac{4(3)}{9.488} \, , \, \frac{4(3)}{.711} \right)$$