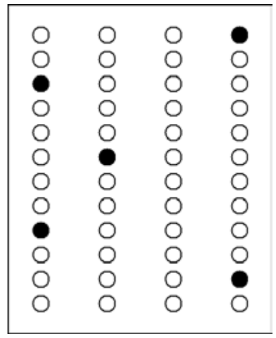
Stat 511 M.S. Exam Spring 2004

ISU engineering students Dodd, Raine, Neaves and Ney experimented on the running of a 48-cavity injection molding machine used to make "pre-forms" for plastic bottles. Pre-forms are produced 48 at a time, in a die with cavities arranged in 4 columns of 12, as shown on the diagram below.



The students weighed pre-forms produced in the 5 cavities indicated by the darkened circles under a number of different circumstances. (Weights were recorded in grams.)

It is desirable that there be no systematic pattern across the die in the weights of pre-forms produced in the different cavities. That is, ideally the mean pre-form weight produced in row r and column c of the die is

$$\mu(r,c) = \mu \tag{1}$$

for some constant μ. A simple possible departure from this might be

$$\mu(r,c) = \beta_0 + \beta_1 r + \beta_2 c \tag{2}$$

for some constants $\beta_0, \beta_1,$ and β_2.

a) Suppose one pre-form is taken from each of 10 consecutive cycles of the machine (under the same machine set-up), all 5 of the test cavities being represented twice. Let

$$y_{rci} = \text{measured mass of the } i\text{th pre-form from row } r \text{ and column } c$$

and

$$\mathbf{Y} = (y_{311}, y_{312}, y_{911}, y_{912}, y_{621}, y_{622}, y_{141}, y_{142}, y_{11,41}, y_{11,42})$$

(for example, $y_{11,41}$ represents the first pre-form from the 11$^{\text{th}}$ row and 4$^{\text{th}}$ column).

Write three sets of assumptions about the mean of $\mathbf{Y}$ in the form $\text{E}\,\mathbf{Y} = \mathbf{X}\boldsymbol{\beta}$ for different sets of $\mathbf{X}$ and $\boldsymbol{\beta}$, representing assumptions (1) and (2), and then "no restrictions on the 5 cavity means."

b) Suppose that error sums of squares for the scenario and 3 models in **a)** are as in the table below

Model for Cavity Means	*SSE*
(1) Constant Mean	.0324
(2) Linear Row and Column Effects	.0126
5 Unrestricted Means	.0045

Give values of F statistics appropriate to testing

i) $H_0 : \beta_1 = \beta_2 = 0$ under assumption (2)

ii) the hypothesis that there is lack of fit to relationship (2)

The R printout attached to this question concerns analyses of weights from 5 consecutive machine cycles (all run under the same machine set-up) where pre-forms from all 5 test cavities were gathered and weighed. With

$$y_{rci} = \text{measured mass of the pre-form from row } r \text{ and column } c \text{ on cycle } i$$

the printout enables inferences under the models

$$y_{rci} = \beta_0 + \beta_1 r + \beta_2 c + \delta_i + \varepsilon_{rci} \tag{3}$$

and

$$y_{rci} = \mu(r,c) + \delta_i + \varepsilon_{rci} \tag{4}$$

where β_0, β_1 and β_2 are constants, the 5 values $\mu(r,c)$ are constants, $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ are iid $\mathrm{N}(0, \sigma_\delta^2)$ independent of the ε_{rci} that are iid $\mathrm{N}(0, \sigma^2)$.

c) Under model (3) is there a statistically significant systematic linear trend in pre-form weight left-to-right across the die? (Explain.)

d) How is it obvious from the printout for model (3) that the size of cycle-to-cycle variability in pre-form weight for a particular cavity is estimated with poor precision?

e) According to the analysis provided for model (4), is there a statistically detectable difference between the mean pre-form weights produced in cavities 18 and 47? (Give approximate 95% confidence limits for this difference.)

f) According to model (4), what is the expected value of the sample variance of the weights of the 5 pre-forms produced in any particular cavity? Give a formula for exact 90% confidence limits for this quantity based on pooling 5 such sample variances.

g) For an (r,c) pair in the data set, let $\bar{y}_{rc.} = \left(\sum_{i=1}^{5} y_{rci}\right)/5$ and y_{rc6} be the pre-form weight produced in row r and column c on a sixth cycle of the machine (immediately after the 5 for which we have data). What are the mean and variance of $y_{rc6} - \bar{y}_{rc.}$? What then is a formula for exact 95% prediction limits for y_{rc6}?

The students did some experimenting with the machine, using as a response variable the average weight of the pre-forms from the 5 cavities indicated on the diagram on page 1 and running various combinations of levels of 3 factors that can be set on the machine, A-Injection Time, B-Hold Time, and C-Hold Pressure.

h) What does model (4) say is the variance of an individual response used in the students' experimentation?

i) The table below gives results from a single cycle for 8 different machine set-ups making up a complete $2\times2\times2$ factorial arrangement in the Factors A,B, and C. If

$$\mu_{ijk} = \text{the mean response at level } i \text{ of A, level } j \text{ of B, and level } k \text{ of C}$$

a sensible definition of the main effect of C at level k is

$$\gamma_k = \mu_{..k} - \mu_{...}$$

Based on the data in the table below, give an estimate of $\gamma_2 - \gamma_1$. Then, based on model (4) and the R printout, give a standard error for your estimate.

Level of A	Level of B	Level of C	Response
1	1	1	22.936
2	1	1	23.000
1	2	1	23.518
2	2	1	23.566
1	1	2	23.272
2	1	2	23.318
1	2	2	23.736
2	2	2	23.794

j) The data in the table above are balanced. Below is an ANOVA table giving some corresponding "sums of squares."

Source	*SS*
A	.00583
B	.54497
C	.15125
AB	.00000
AC	.00001
BC	.00541
ABC	.00010

Define the 8×1 vectors $\mathbf{Y},\mathbf{1},\mathbf{X}_{\text{B}},\mathbf{X}_{\text{C}}$, and $\mathbf{X}_{\text{BC}}$ and the 8×4 matrix $\mathbf{X}$ by

$$\mathbf{Y}=\begin{pmatrix}22.936\\23.000\\23.518\\23.566\\23.272\\23.318\\23.736\\23.794\end{pmatrix},\mathbf{1}=\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix},\mathbf{X}_{\text{B}}=\begin{pmatrix}-1\\-1\\1\\1\\-1\\-1\\1\\1\end{pmatrix},\mathbf{X}_{\text{C}}=\begin{pmatrix}-1\\-1\\-1\\-1\\1\\1\\1\\1\end{pmatrix},\mathbf{X}_{\text{BC}}=\begin{pmatrix}1\\1\\-1\\-1\\-1\\-1\\1\\1\end{pmatrix}, \text{ and } \mathbf{X}=\left(\mathbf{1}\,|\,\mathbf{X}_{\text{B}}\,|\,\mathbf{X}_{\text{C}}\,|\,\mathbf{X}_{\text{BC}}\right)$$

Let $\mathbf{P}_{\mathbf{X}}$ be the perpendicular projection matrix onto $C(\mathbf{X})$, the column space of $\mathbf{X}$. Evaluate the quadratic form

$$\mathbf{Y}'\left(\mathbf{I}-\mathbf{P}_{\mathbf{X}}\right)\mathbf{Y}$$

Stat 511 Printout

```
> CAVITY<-as.factor(cavity)
> Data<-data.frame(y,CAVITY,row,col,cycle)
> Data
       y CAVITY row col cycle
1  23.36      3   3   1     1
2  23.37      3   3   1     2
3  23.33      3   3   1     3
4  23.34      3   3   1     4
5  23.35      3   3   1     5
6  23.36      9   9   1     1
7  23.36      9   9   1     2
8  23.33      9   9   1     3
9  23.34      9   9   1     4
10 23.35      9   9   1     5
11 23.19     18   6   2     1
12 23.20     18   6   2     2
13 23.28     18   6   2     3
14 23.29     18   6   2     4
15 23.23     18   6   2     5
16 23.30     37   1   4     1
17 23.32     37   1   4     2
18 23.32     37   1   4     3
19 23.34     37   1   4     4
20 23.32     37   1   4     5
21 23.38     47  11   4     1
22 23.39     47  11   4     2
23 23.37     47  11   4     3
24 23.36     47  11   4     4
25 23.37     47  11   4     5

> output1<-lme(y~1+row+col,random=~1|cycle)

> summary(output1)
Linear mixed-effects model fit by REML
 Data: NULL
        AIC       BIC   logLik
  -44.19070 -38.73549 27.09535

Random effects:
 Formula: ~1 | cycle
         (Intercept)   Residual
StdDev: 0.0004287742 0.05269565

Fixed effects: y ~ 1 + row + col
                Value   Std.Error DF  t-value p-value
(Intercept) 23.292793 0.027437839 18 848.9296  0.0000
row          0.003882 0.002857823 18   1.3585  0.1911
col          0.004130 0.007769549 18   0.5316  0.6015
 Correlation:
    (Intr) row
row -0.625
col -0.680  0.000
```

```
Standardized Within-Group Residuals:
        Min          Q1         Med          Q3         Max 
-2.54945497 -0.03538148  0.34119592  0.53391294  1.16573207 

Number of Observations: 25
Number of Groups: 5 

> intervals(output1)
Approximate 95% confidence intervals

 Fixed effects:
                  lower         est.        upper
(Intercept) 23.235148079 23.292792839 23.350437599
row         -0.002121710  0.003882353  0.009886416
col         -0.012192783  0.004130435  0.020453652
attr(,"label")
[1] "Fixed effects:"

 Random Effects:
  Level: cycle 
                      lower         est.        upper
sd((Intercept)) 1.816675e-21 0.0004287742 1.011999e+14

 Within-group standard error:
     lower       est.      upper 
0.03921486 0.05269565 0.07081068 

> predict(output1)
       1        2        3        4        5        1        2        3 
23.30857 23.30857 23.30857 23.30857 23.30857 23.33186 23.33187 23.33186 
       4        5        1        2        3        4        5        1 
23.33187 23.33186 23.32435 23.32435 23.32435 23.32435 23.32435 23.31319 
       2        3        4        5        1        2        3        4 
23.31320 23.31320 23.31320 23.31320 23.35202 23.35202 23.35202 23.35202 
       5 
23.35202 
attr(,"label")
[1] "Fitted values"

> vcov(output1)
             (Intercept)           row           col
(Intercept)  0.0007528350 -4.900290e-05 -1.448781e-04
row         -0.0000490029  8.167151e-06 -1.867372e-21
col         -0.0001448781 -1.867372e-21  6.036590e-05
```

```
> output2<-lme(y~CAVITY-1,random=~1|cycle)

> summary(output2)
Linear mixed-effects model fit by REML
 Data: NULL
        AIC       BIC   logLik
  -70.80317 -63.83304 42.40158

Random effects:
 Formula: ~1 | cycle
         (Intercept)   Residual
StdDev: 0.0002226945 0.02374837

Fixed effects: y ~ CAVITY - 1
          Value  Std.Error DF  t-value p-value
CAVITY3  23.350 0.01062106 16 2198.462       0
CAVITY9  23.348 0.01062106 16 2198.274       0
CAVITY18 23.238 0.01062106 16 2187.917       0
CAVITY37 23.320 0.01062106 16 2195.638       0
CAVITY47 23.374 0.01062106 16 2200.722       0
 Correlation:
         CAVITY3 CAVITY9 CAVITY1 CAVITY37
CAVITY9  0
CAVITY18 0       0
CAVITY37 0       0       0
CAVITY47 0       0       0       0

Standardized Within-Group Residuals:
          Min            Q1           Med            Q3           Max
-2.021043e+00 -4.212296e-01  1.495982e-13  5.052608e-01  2.189476e+00

Number of Observations: 25
Number of Groups: 5

> intervals(output2)
Approximate 95% confidence intervals

 Fixed effects:
            lower   est.    upper
CAVITY3  23.32748 23.350 23.37252
CAVITY9  23.32548 23.348 23.37052
CAVITY18 23.21548 23.238 23.26052
CAVITY37 23.29748 23.320 23.34252
CAVITY47 23.35148 23.374 23.39652
attr(,"label")
[1] "Fixed effects:"
```

```
 Random Effects:
  Level: cycle
                       lower         est.        upper
sd((Intercept)) 2.002516e-23 0.0002226945 2.476527e+15

 Within-group standard error:
     lower       est.      upper
0.01741993 0.02374837 0.03237585

> predict(output2)
       1        2        3        4        5        1        2        3
23.35000 23.35000 23.35000 23.35000 23.35000 23.34800 23.34800 23.34800
       4        5        1        2        3        4        5        1
23.34800 23.34800 23.23800 23.23800 23.23800 23.23800 23.23800 23.32000
       2        3        4        5        1        2        3        4
23.32000 23.32000 23.32000 23.32000 23.37400 23.37400 23.37400 23.37400
       5
23.37400
attr(,"label")
[1] "Fitted values"

> vcov(output2)
              CAVITY3      CAVITY9     CAVITY18     CAVITY37     CAVITY47
CAVITY3  1.128069e-04 9.918572e-09 9.918572e-09 9.918572e-09 9.918572e-09
CAVITY9  9.918572e-09 1.128069e-04 9.918572e-09 9.918572e-09 9.918572e-09
CAVITY18 9.918572e-09 9.918572e-09 1.128069e-04 9.918572e-09 9.918572e-09
CAVITY37 9.918572e-09 9.918572e-09 9.918572e-09 1.128069e-04 9.918572e-09
CAVITY47 9.918572e-09 9.918572e-09 9.918572e-09 9.918572e-09 1.128069e-04
```