Summary of Some Useful Facts About Multivariate Normal Distributions

1. The MVN distribution is most usefully defined as the distribution of $\underset{k\times 1}{X} = \underset{k\times p}{A}\,\underset{p\times 1}{Z} + \underset{k\times 1}{m}$ for $\underset{p\times 1}{Z}$ a vector of independent standard normal random variables. Such a random vector has mean vector $\underset{k\times 1}{m}$ and covariance matrix $\underset{k\times k}{\Sigma} = \underset{k\times p}{A} \times \underset{p\times k}{A'}$. (This definition turns out to be unambiguous … any dimension p and any matrix A giving a particular Σ end up producing the same k-dimensional joint distribution.)
2. (Not explicitly said in class … only the MVN version was stated) If $\underset{k\times 1}{X}$ has mean vector $\underset{k\times 1}{m}$ and covariance matrix $\underset{k\times k}{\Sigma}$, then $\underset{l\times 1}{Y} = \underset{l\times k}{B}\,\underset{k\times 1}{X} + \underset{l\times 1}{d}$ has mean vector $\underset{l\times k}{B}\,\underset{k\times 1}{m} + \underset{l\times 1}{d}$ and covariance matrix $\underset{l\times k}{B}\,\underset{k\times k}{\Sigma}\,\underset{k\times l}{B'}$.
3. If $\underset{k\times 1}{X}$ is $\text{MVN}_k\left(\underset{k\times 1}{m}, \underset{k\times k}{\Sigma}\right)$ then $\underset{l\times 1}{Y} = \underset{l\times k}{B}\,\underset{k\times 1}{X} + \underset{l\times 1}{d}$ is $\text{MVN}_l\left(\underset{l\times k}{B}\,\underset{k\times 1}{m} + \underset{l\times 1}{d}, \underset{l\times k}{B}\,\underset{k\times k}{\Sigma}\,\underset{k\times l}{B'}\right)$.
4. If X is MVN_k, its individual marginal distributions are univariate normal. Further, any sub-vector of dimension $l < k$ is MVN_l (with mean vector the appropriate sub-vector of $\underset{k\times 1}{m}$ and covariance matrix the appropriate sub-matrix of $\underset{k\times k}{\Sigma}$).
5. If $\underset{k\times 1}{X}$ is $\text{MVN}_k\left(\underset{k\times 1}{m_1}, \underset{k\times k}{\Sigma_{11}}\right)$ and independent of $\underset{l\times 1}{Y}$ which is $\text{MVN}_l\left(\underset{l\times 1}{m_2}, \underset{l\times l}{\Sigma_{22}}\right)$, then the vector
$$\underset{(k+l)\times 1}{W} = \begin{pmatrix} X \\ Y \end{pmatrix} \text{is } \text{MVN}_{k+l}\left(\begin{pmatrix} \underset{k\times 1}{m_1} \\ \underset{l\times 1}{m_2} \end{pmatrix}, \begin{pmatrix} \underset{k\times k}{\Sigma_{11}} & \underset{k\times l}{0} \\ \underset{l\times k}{0} & \underset{l\times l}{\Sigma_{22}} \end{pmatrix}\right).$$
6. For non-singular $\underset{k\times k}{\Sigma}$ the $\text{MVN}_k\left(\underset{k\times 1}{m}, \underset{k\times k}{\Sigma}\right)$ distribution has a (joint) pdf on k-dimensional space given by
$$f_X(x) = (2p)^{-\frac{k}{2}} \left|\det\Sigma\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)\right)$$
7. The joint pdf given in 6 above can be studied and conditional distributions (given values for part of the X vector) identified. For $\underset{k\times 1}{X} = \begin{pmatrix} \underset{l\times 1}{X_1} \\ \underset{(k-l)\times 1}{X_2} \end{pmatrix}$ $\text{MVN}_k\left(\underset{k\times 1}{m}, \underset{k\times k}{\Sigma}\right)$ where
$$\underset{k\times 1}{m} = \begin{pmatrix} \underset{l\times 1}{m_1} \\ \underset{(k-l)\times 1}{m_2} \end{pmatrix} \text{and } \underset{k\times k}{\Sigma} = \begin{pmatrix} \underset{l\times l}{\Sigma_{11}} & \underset{l\times (k-l)}{\Sigma_{12}} \\ \underset{(k-l)\times l}{\Sigma_{21}} & \underset{(k-l)\times(k-l)}{\Sigma_{22}} \end{pmatrix}$$
the conditional distribution of X_1 given that $X_2 = x_2$ is MVN_l with
$$\text{mean vector} = m_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - m_2)$$
and
$$\text{covariance matrix} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$
8. All correlations between two parts of a MVN vector equal to 0 implies that those parts of the vector are independent.