STAT 447 HW 8 Spring 2002 Key

7.3 $P(|\bar{Y}-\mu| \le 2) = P(-2 \le (\bar{Y}-\mu) \le 2) = P\left(-\frac{2}{\sigma/\sqrt{n}} \le \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \le \frac{2}{\sigma/\sqrt{n}}\right)$

$= P\left(-\frac{2}{4/\sqrt{9}} \le z \le \frac{2}{4/\sqrt{9}}\right) = P(-1.5 \le z \le 1.5)$

$= 1 - 2P(z > 1.5) = 1 - 2\times 0.0668 = 0.8664$

7.4 $P(|\bar{Y}-\mu| \le 1) = 0.90$.

$\Rightarrow P(-1 \le \bar{Y}-\mu \le 1) = P\left(-\frac{1}{\sigma/\sqrt{n}} \le z \le \frac{1}{\sigma/\sqrt{n}}\right) = P\left(-\frac{\sqrt{n}}{4} \le z \le \frac{\sqrt{n}}{4}\right)$

$= 0.90$

$\Rightarrow \frac{\sqrt{n}}{4} = 1.645$

$\Rightarrow n = 43.30$

we need 44 trees

7.7 a. $E(\bar{X}-\bar{Y}) = E(\bar{X}) - E(\bar{Y}) = \mu_1 - \mu_2$

b. $V(\bar{X}-\bar{Y}) = V(\bar{X}) + V(\bar{Y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$

c. $P(|(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)| \le 1) = 0.95$

$\Rightarrow P\left(\frac{|(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)|}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}} \le \frac{1}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{m}}}\right)$

$= P\left(z \le \frac{1}{\sqrt{\frac{2}{n}+\frac{2.5}{n}}}\right) = P\left(z \le \sqrt{\frac{n}{4.5}}\right) = 0.95$

$\Rightarrow \sqrt{\frac{n}{4.5}} = 1.96$

$\Rightarrow n = 17.29$

We need sample size $m = n = 18$

7.8 $P[(\bar{X}_A - \bar{X}_B) \ge 1] = P\left[\frac{(\bar{X}_A-\bar{X}_B)-(\mu_A-\mu_B)}{\sqrt{\frac{\sigma_A^2}{n}+\frac{\sigma_B^2}{n}}} \ge \frac{1-0}{\sqrt{\frac{0.4}{10}+\frac{0.8}{10}}}\right]$

$= P(z \ge 2.89)$

$= 0.0019$

7.11 $n = 20 \quad \sigma^2 = 1.4$

a. $P(S^2 \le b) = P\left(\frac{(n-1)S^2}{\sigma^2} \le \frac{(n-1)b}{\sigma^2}\right) = P\left(\chi^2_{(n-1)} \le \frac{(n-1)b}{\sigma^2}\right) = 0.975$

$\frac{(n-1)b}{\sigma^2} = \chi^2_{0.025,\,19} = 32.8523$

$\Rightarrow b = 2.42$

b. $P(a \le S^2) = P\left(\frac{(n-1)a}{\sigma^2} \le \frac{(n-1)S^2}{\sigma^2}\right) = P\left(\frac{(n-1)a}{\sigma^2} \le \chi^2_{(n-1)}\right) = 0.975$

$\frac{(n-1)a}{\sigma^2} = \chi^2_{0.975,\,19} = 8.90655$

$\Rightarrow a = 0.656$

c. $P(a \le S^2 \le b) = P(a \le S^2) - P(S^2 > b)$

$= P(a \le S^2) - (1 - P(S^2 \le b))$

$= 0.975 - (1 - 0.975) = 0.95$

7.15 $T = \frac{Z}{\sqrt{W/\nu}}$ $Z \sim N(0,1)$ $W \sim \chi^2_\nu$

$F = \frac{W_1/\nu_1}{W_2/\nu_2}$ $W_1 \sim \chi^2_{\nu_1}$ W_1 and W_2 are independent

$W_2 \sim \chi^2_{\nu_2}$

$\Rightarrow U = T^2 = \frac{Z^2}{W/\nu} = \frac{Z^2/1}{W/\nu}$

By theorem 7.2 $Z^2 \sim \chi^2_1$

By Definition 7.3, $U \sim F_{1,\nu}$

7.18 $\sigma_1^2 = 2\sigma_2^2$ $n_1 = 10$ $n_2 = 8$

a. $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2/2\sigma_2^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{2S_2^2} \sim F_{9,7}$

$P(\frac{S_1^2}{S_2^2} \le b) = P(\frac{S_1^2}{2S_2^2} \le \frac{b}{2}) = 0.95$

$\Rightarrow P(\frac{S_1^2}{2S_2^2} > \frac{b}{2}) = 0.05$

$\Rightarrow \frac{b}{2} = F_{0.05,9,7} = 3.68$

$\Rightarrow b = 7.36$

b. $P(a \le \frac{S_1^2}{S_2^2}) = P(\frac{a}{2} \le \frac{S_1^2}{2S_2^2}) = 0.95$

$\Rightarrow P(\frac{2S_2^2}{S_1} \le \frac{2}{a}) = 0.95$

$\Rightarrow P(\frac{2S_2^2}{S_1} > \frac{2}{a}) = 0.05$

$\Rightarrow \frac{2}{a} = F_{0.05,7,9} = 3.29$

$\Rightarrow a = 0.608$

c. $P(a \le \frac{S_1^2}{S_2^2} \le b) = P(a \le \frac{S_1^2}{S_2^2}) - P(\frac{S_1^2}{S_2^2} > b)$

$= P(a \le \frac{S_1^2}{S_2^2}) - (1 - P(\frac{S_1^2}{S_2^2} \le b))$

$= 0.95 - (1 - 0.95) = 0.90$

7.21 a. $\bar{X}_i \sim N(\mu_i, \sigma^2/n_i)$ $\bar{X}_i$ are independent

By theorem 6.3, $\hat{\theta} = \sum_{i=1}^{k} c_i \bar{X}_i \sim N(\sum_{i=1}^{k} c_i\mu_i, \sum_{i=1}^{k} c_i^2\sigma^2/n_i)$

b. $(n_i - 1) S_i^2/\sigma^2 \sim \chi^2_{(n_i-1)}$ (Theorem 7.3)

$(n_i - 1) S_i^2/\sigma^2$ are independent

The sum of χ^2 random variables is χ^2 with degrees equal to the sum of degrees of freedom of the summands.

Hence $SSE/\sigma^2 = \sum_{i=1}^{k} (n_i - 1) S_i^2/\sigma^2 \sim \chi^2_{(\sum_{i=1}^{k}(n_i-1))} = \chi^2_{(\sum_{i=1}^{k} n_i - k)}$

c. $t = \frac{\hat{\theta} - \theta}{\sqrt{MSE \sum_{i=1}^{k} \frac{c_i^2}{n_i}}} = \frac{\hat{\theta} - \theta}{\sqrt{\frac{SSE}{(\sum_{i=1}^{k} n_i - k)\sigma^2} \sum_{i=1}^{k} \frac{c_i^2\sigma^2}{n_i}}} = \frac{\hat{\theta} - \theta}{\sqrt{\sum_{i=1}^{k} \frac{c_i^2\sigma^2}{n_i}}} \Big/ \sqrt{\frac{SSE}{(\sum_{i=1}^{k} n_i - k)\sigma^2}}$

The numerator follows standard normal distribution

The denominator is the square root of $\chi^2_{(\sum_{i=1}^{k} n_i - k)}/(\sum_{i=1}^{k} n_i - k)$

SSE and $\hat{\theta}$ are independent since S_i^2 and $\bar{X}_i$ are independent and S_j^2 and $\bar{X}_i$ are independent for $i \ne j$

$\Rightarrow t \sim t_{(\sum_{i=1}^{k} n_i - k)}$

7.25 $\mu = 7.00$ $\sigma = 0.50$ $n = 64$

$P(\bar{X} \le 6.90) = P[Z \le (6.90 - 7.00)/(0.50/\sqrt{64})] = P(Z \le -1.6) = 0.0548$

It is reasonable to assume the wage rates are equal.

7.26 $n = 40$, $\sigma \approx$ range$/4 = 3/4$

$P(|\bar{x} - \mu| \le 0.2) = P(-0.2 \le (\bar{x} - \mu) \le 0.2)$

$= P\left(-\frac{0.2}{\frac{3}{4}/\sqrt{40}} \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le \frac{0.2}{\frac{3}{4}/\sqrt{40}}\right) = P(-1.69 \le z \le 1.69)$

$= 1 - 2 \times 0.0455 = 0.909$

7.27 $P(|\bar{x} - \mu| \le 0.1) = P\left(|z| \le \frac{0.1\sqrt{n}}{\frac{3}{4}}\right) = 0.90$

$\Rightarrow \frac{0.1\sqrt{n}}{3/4} = 1.645$

$\Rightarrow n = 152.21$

The scientist should take 153 samples.

7.42 $P\left(\sum_{1}^{100} x_i > 240\right) = P(\bar{x} > 2.4) = P\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} > \frac{2.4 - 2.5}{2/\sqrt{100}}\right)$

$= P(z > -0.5) = 1 - 0.3085 = 0.6915$

7.75. $P(X \le 110) = P\left(\frac{X-\lambda}{\sqrt{\lambda}} \le \frac{110-\lambda}{\sqrt{\lambda}}\right) = P\left(z \le \frac{110-100}{\sqrt{100}}\right) = P(z \le 1)$

$= 1 - 0.1587 = 0.8413$

8.4 $Y_i \sim$ Exponential (θ) $E(Y_i) = \theta$ $V(Y_i) = \theta^2$

a. $E(\hat{\theta}_1) = E(Y_1) = \theta$

$E(\hat{\theta}_2) = E\left(\frac{Y_1+Y_2}{2}\right) = \theta$

$E(\hat{\theta}_3) = E\left(\frac{Y_1+2Y_2}{3}\right) = \theta$

$E(\hat{\theta}_5) = E(\bar{Y}) = \theta$

$\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, and $\hat{\theta}_4$ are unbiased estimators for θ

$\hat{\theta}_4 = \min(Y_1, Y_2, Y_3) = Y_{(1)}$ $F(y) = 1 - e^{-y/\theta}$

$f_{(1)}(y) = \frac{3!}{(3-1)!}[1 - F(y)]^{3-1} f(y)$

$= 3(e^{-y/\theta})^2 \cdot \frac{1}{\theta} e^{-y/\theta} = \frac{3}{\theta} e^{-3y/\theta}$

$Y_{(1)} \sim$ Exponential $\left(\frac{\theta}{3}\right)$

$E(\hat{\theta}_4) = E(Y_{(1)}) = \frac{\theta}{3} \Rightarrow \hat{\theta}_4$ is biased estimator for θ

b. $V(\hat{\theta}_1) = V(Y_1) = \theta^2$

$V(\hat{\theta}_2) = V\left(\frac{Y_1+Y_2}{2}\right) = \frac{1}{4}(V(Y_1) + V(Y_2)) = \frac{\theta^2}{2}$

$V(\hat{\theta}_3) = V\left(\frac{Y_1+2Y_2}{3}\right) = \frac{1}{9}V(Y_1) + \frac{4}{9}V(Y_2) = \frac{5}{9}\theta^2$

$V(\hat{\theta}_5) = V(\bar{Y}) = \frac{1}{9}[V(Y_1) + V(Y_2) + V(Y_3)] = \frac{1}{3}\theta^2$

$\Rightarrow \hat{\theta}_5$ has the smallest variables

8.10 $f(y) = \alpha y^{\alpha-1}/\theta^{\alpha}$, $0 \le y \le \theta$, $F(y) = \int_0^y \alpha t^{\alpha-1}/\theta^{\alpha}\, dt = (\frac{y}{\theta})^{\alpha}$

$f_{(n)}(y) = \frac{n!}{(n-1)!}[(\frac{y}{\theta})^{\alpha}]^{n-1} \alpha y^{\alpha-1}/\theta^{\alpha} = n\alpha y^{n\alpha-1}/\theta^{n\alpha}$ $\quad 0 \le y \le \theta$

a. $E(\hat{\theta}) = E(Y_{(n)}) = \int_0^{\theta} y\, n\alpha\, y^{n\alpha-1}/\theta^{n\alpha}\, dy$

$= \frac{1}{\theta^{n\alpha}} \frac{n\alpha}{n\alpha+1} y^{n\alpha+1} \Big|_0^{\theta} = \frac{n\alpha}{n\alpha+1}\theta \neq \theta$

b. $\frac{n\alpha+1}{n\alpha}\hat{\theta} = \frac{n\alpha+1}{n\alpha} Y_{(n)}$ is unbiased

c. $MSE(\hat{\theta}) = E[(Y_{(n)} - \theta)^2] = E(Y_{(n)}^2) - 2\theta E(Y_{(n)}) + \theta^2$

$E(Y_{(n)}^2) = \int_0^{\theta} y^2\, n\alpha\, y^{n\alpha-1}/\theta^{n\alpha}\, dy$

$= \frac{1}{\theta^{n\alpha}} \frac{n\alpha}{n\alpha+2} y^{n\alpha+2} \Big|_0^{\theta} = \frac{n\alpha}{n\alpha+2}\theta^2$

$\Rightarrow MSE(\hat{\theta}) = \frac{n\alpha}{n\alpha+2}\theta^2 - 2\theta \cdot \frac{n\alpha}{n\alpha+1}\theta + \theta^2$

$= \frac{2\theta^2}{(n\alpha+1)(n\alpha+2)}$

8.12. $S = \sqrt{S^2}$ $\quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Let $X = \frac{(n-1)S^2}{\sigma^2}$

$f(x) = \frac{x^{[(n-1)/2]-1} e^{-x/2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}}$ $\qquad Y = S^2 = \frac{\sigma^2 X}{n-1}$

$g(y) = f(\frac{(n-1)y}{\sigma^2}) \left|\frac{dx}{dy}\right| = \frac{(\frac{n-1}{\sigma^2})^{(n-1)/2} y^{[(n-1)/2]-1} e^{-(n-1)y/2\sigma^2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}}$

a. $E(S) = E(\sqrt{Y}) = \int_0^{\infty} y^{1/2} g(y)\, dy$

$= \int_0^{\infty} y^{1/2} \frac{(\frac{n-1}{\sigma^2})^{(n-1)/2} y^{[(n-1)/2]-1} e^{-(n-1)y/2\sigma^2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2}}\, dy$

$= \frac{(\frac{n-1}{\sigma^2})^{(n-1)/2} \Gamma(\frac{n}{2})(2\sigma^2)^{n/2}}{\Gamma(\frac{n-1}{2}) 2^{(n-1)/2} (n-1)^{n/2}} \int_0^{\infty} \frac{y^{(n/2)-1} e^{-(n-1)y/2\sigma^2}}{\Gamma(\frac{n}{2})(\frac{2\sigma^2}{n-1})^{n/2}}\, dy$

$= \frac{(n-1)^{-\frac{1}{2}} \Gamma(\frac{n}{2})(\sigma^2)^{1/2}}{\Gamma(\frac{n-1}{2}) 2^{-1/2}} = \frac{\Gamma(\frac{n}{2}) 2^{1/2} \sigma}{\Gamma(\frac{n-1}{2})\sqrt{n-1}} \neq \sigma$

b. $\hat{\sigma} = S \frac{\Gamma(\frac{n-1}{2})\sqrt{n-1}}{\Gamma(\frac{n}{2})\sqrt{2}}$ is unbiased

c. $E(\bar{Y}) = \mu$

Thus $\bar{Y} - z_{\alpha}\hat{\sigma}$ is an unbiased estimator of $\mu - z_{\alpha}\sigma$