6.1. a. $F_{U_1}(u) = P(U_1 \le u) = P(2Y-1 \le u) = P(Y \le \frac{u+1}{2}) = F_Y(\frac{u+1}{2})$

$$F_Y(y) = \begin{cases} \int_0^y 2(1-t)\,dt = 2y - y^2 & 0 \le y \le 1 \\ 0 & y < 0 \\ 1 & y > 1 \end{cases}$$

$$F_{U_1}(u) = \begin{cases} 2(\frac{u+1}{2}) - (\frac{u+1}{2})^2 & 0 \le \frac{u+1}{2} \le 1 \quad \text{i.e. } -1 \le u \le 1 \\ 0 & \frac{u+1}{2} < 0 \quad \text{i.e. } u < -1 \\ 1 & \frac{u+1}{2} > 1 \quad \text{i.e. } u > 1 \end{cases}$$

$$\Rightarrow f_{U_1}(u) = F'_{U_1}(u) = \begin{cases} 1 - \frac{1}{4} \cdot 2(u+1) = \frac{1-u}{2} & -1 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$$

b. $F_{U_2}(u) = P(U_2 \le u) = P(1-2Y \le u) = P(Y \ge \frac{1-u}{2}) = 1 - F_Y(\frac{1-u}{2})$

$$F_{U_2}(u) = \begin{cases} 1 - 2(\frac{1-u}{2}) + (\frac{1-u}{2})^2 & 0 \le \frac{1-u}{2} \le 1 \quad \text{i.e. } -1 \le u \le 1 \\ 1 & \frac{1-u}{2} < 0 \quad \text{i.e. } u > 1 \\ 0 & \frac{1-u}{2} > 1 \quad \text{i.e. } u < -1 \end{cases}$$

$$\Rightarrow f_{U_2}(u) = F'_{U_2}(u) = \begin{cases} 1 + \frac{1}{4} \cdot 2(u-1) = \frac{u+1}{2} & -1 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$$

c. $F_{U_3}(u) = P(U_3 \le u) = P(Y^2 \le u) = P(Y \le \sqrt{u}) = F_Y(\sqrt{u})$

$$F_{U_3}(u) = \begin{cases} 2\sqrt{u} - u & 0 \le u < 1 \\ 0 & u < 0 \\ 1 & u > 1 \end{cases}$$

$$\Rightarrow f_{U_3}(u) = F'_{U_3}(u) = \begin{cases} \frac{1}{\sqrt{u}} - 1 & 0 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$$

d. $E(U_1) = \int_{-1}^{1} u \cdot \frac{1-u}{2}\,du = [\frac{1}{4}u^2 - \frac{1}{6}u^3]_{-1}^{1} = -\frac{1}{3}$

$E(U_2) = \int_{-1}^{1} u \, \frac{u+1}{2}\,du = [\frac{1}{4}u^2 + \frac{1}{6}u^3]_{-1}^{1} = \frac{1}{3}$

$E(U_3) = \int_0^1 u \cdot (\frac{1}{\sqrt{u}} - 1)\,du = [\frac{2}{3}u^{\frac{3}{2}} - \frac{1}{2}u^2]_0^1 = \frac{1}{6}$

e. $E(Y) = \int_0^1 2(1-y)\,y\,dy = [y^2 - \frac{2}{3}y^3]_0^1 = \frac{1}{3}$

$E(Y^2) = \int_0^1 2(1-y)\,y^2\,dy = [\frac{2}{3}y^3 - \frac{1}{2}y^4]_0^1 = \frac{1}{6}$

$E(U_1) = E(2Y-1) = 2E(Y) - 1 = 2 \times \frac{1}{3} - 1 = -\frac{1}{3}$

$E(U_2) = E(1-2Y) = 1 - 2E(Y) = 1 - 2 \times \frac{1}{3} = \frac{1}{3}$

$E(U_3) = E(Y^2) = \frac{1}{6}$

6.2 a. $F_Y(y) = \int_{-1}^{y} \frac{3}{2}t^2\,dt = \frac{1}{2}(y^3 + 1) \qquad -1 \le y \le 1$

$F_{U_1}(u) = P(3Y \le u) = P(Y \le \frac{u}{3}) = F_Y(\frac{u}{3})$

$$F_{U_1}(u) = \begin{cases} \frac{1}{2}\{(\frac{u}{3})^3 + 1\} & -1 \le \frac{u}{3} \le 1 \quad \text{i.e. } -3 \le u \le 3 \\ 0 & \frac{u}{3} < -1 \quad \text{i.e. } u < -3 \\ 1 & \frac{u}{3} > 1 \quad \text{i.e. } u > 3 \end{cases}$$

$\Rightarrow f_{U_1}(u) = F'_{U_1}(u) = \begin{cases} \frac{u^2}{18} & -3 \le u \le 3 \\ 0 & o.w. \end{cases}$

b. $F_{U_2}(u) = P(3-Y \le u) = P(Y \ge 3-u) = 1 - F_Y(3-u)$

$F_{U_2}(u) = \begin{cases} 1 - \frac{1}{2}\{(3-u)^3+1\} & -1 \le 3-u \le 1 \quad \text{i.e} \quad 2 \le u \le 4 \\ 1 & 3-u < -1 \quad \text{i.e} \quad u > 4 \\ 0 & 3-u > 1 \quad \text{i.e} \quad u < 2 \end{cases}$

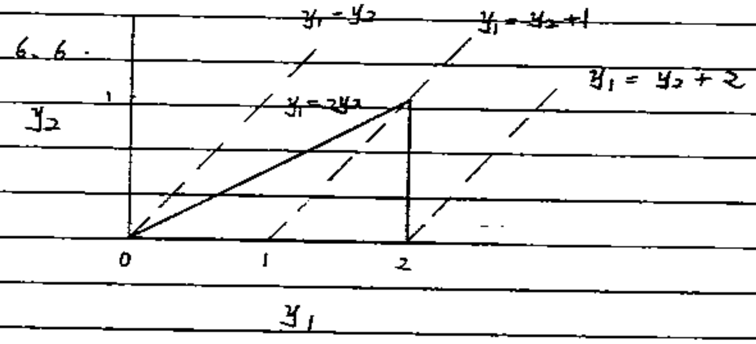
$\Rightarrow f_{U_2}(u) = F'_{U_2}(u) = \begin{cases} \frac{3}{2}(3-u)^2 & 2 \le u \le 4 \\ 0 & o.w. \end{cases}$

C. $F_{U_3}(u) = P(Y^2 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u}) = F_Y(\sqrt{u}) - F_Y(-\sqrt{u})$

$F_{U_3}(u) = \begin{cases} \frac{1}{2}(u^{\frac{3}{2}} + 1) - \frac{1}{2}(-u^{\frac{3}{2}} + 1) = u^{\frac{3}{2}} & 0 \le u \le 1 \\ 0 & u < 0 \\ 1 & u > 1 \end{cases}$

$\Rightarrow f_{U_3}(u) = F'_{U_3}(u) = \begin{cases} \frac{3}{2}\sqrt{u} & 0 \le u \le 1 \\ 0 & o.w. \end{cases}$

6.6.



$U = Y_1 - Y_2$

a. For $u \le 0$ $\quad F_U(u) = P(Y_1 - Y_2 \le u) = 0$

For $0 \le u \le 1$ $\quad F_U(u) = \int_0^u \int_{2y_2}^{y_2+u} dy_1\, dy_2 = \int_0^u (u - y_2)\, dy_2 = \frac{u^2}{2}$

For $1 \le u \le 2$ $\quad F_U(u) = 1 - \int_0^{2-u} (2 - y_2 - u)\, dy_2 = 1 - \frac{(2-u)^2}{2}$

For $u > 2$ $\quad F_U(u) = 1$

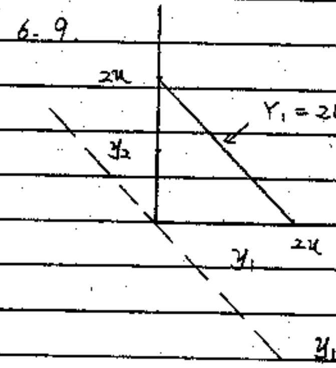
$\Rightarrow f_U(u) = F'_U(u) = \begin{cases} u & 0 \le u \le 1 \\ 2-u & 1 \le u \le 2 \\ 0 & o.w. \end{cases}$

b. $E(U) = \int_0^1 u \cdot u\, du + \int_1^2 u(2-u)\, du = \frac{1}{3}u^3 \Big|_0^1 + [u^2 - \frac{1}{3}u^3]_1^2$

$= \frac{1}{3} + 4 - 1 - \frac{8}{3} + \frac{1}{3} = 1$

Same as in 5.66 (C)

6.9



$Y_1, Y_2 \sim \exp(1)$

$f(y_i) = e^{-y_i} \quad y_i > 0, \quad i = 1, 2$

$f(y_1, y_2) = f(y_1) \cdot f(y_2) = e^{-(y_1+y_2)} \quad y_1 > 0, \quad y_2 > 0$

a. Let $u = \frac{Y_1+Y_2}{2}$

$$F_U(u) = P\left(\frac{Y_1+Y_2}{2} \le u\right) = P(Y_1 + Y_2 \le 2u)$$
$$= \int_0^{2u} \int_0^{2u-y_2} e^{-(y_1+y_2)}\, dy_1\, dy_2$$
$$= \int_0^{2u} (e^{-y_2} - e^{-2u})\, dy_2 = \left[-e^{-y_2} - y_2 e^{-2u}\right]_0^{2u}$$
$$= -e^{-2u} + 1 - 2u e^{-2u} = 1 - e^{-2u} - 2u e^{-2u} \quad \text{for } u \ge 0$$
$$\Rightarrow f_U(u) = F'_U(u) = \begin{cases} 2e^{-2u} - 2e^{-2u} + 4u e^{-2u} = 4u e^{-2u} & \text{for } u \ge 0 \\ 0 & \text{o.w.} \end{cases}$$

$U \sim \text{Gamma}(2, \frac{1}{2})$

b. $E(U) = \alpha\beta = 2 \cdot \frac{1}{2} = 1$

$V(U) = \alpha\beta^2 = 2 \cdot (\frac{1}{2})^2 = \frac{1}{2}$

$E(U) = E\left(\frac{Y_1+Y_2}{2}\right) = \frac{1}{2}E(Y_1) + \frac{1}{2}E(Y_2) = \frac{1}{2} + \frac{1}{2} = 1$

$V(U) = \frac{1}{4}V(Y_1) + \frac{1}{4}V(Y_2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

6.11 $F(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y^2} & y \ge 0 \end{cases}$

$U \sim \text{Uniform}(0, 1)$

$F(u) = \begin{cases} u & 0 \le u \le 1 \\ 0 & u < 0 \\ 1 & u > 1 \end{cases}$

$F(G(u)) = \begin{cases} 0 & G(u) < 0 \\ 1 - e^{-G(u)^2} & G(u) \ge 0 \end{cases}$

$1 - e^{-G(u)^2} = u$

$G(u) = [-\ln(1-u)]^{1/2}$

6.19 a. $U_1 = 2Y - 1 \quad Y = \frac{u+1}{2} \quad \frac{dy}{du} = \frac{1}{2}$

$f_{U_1}(u) = \frac{1}{2} \cdot 2\left(1 - \frac{u+1}{2}\right) = \frac{1-u}{2} \quad 0 \le \frac{u+1}{2} \le 1 \quad$ i.e. $-1 \le u \le 1$

b. $U_2 = 1 - 2Y \quad Y = \frac{1-u}{2} \quad \frac{dy}{du} = -\frac{1}{2}$

$f_{U_2}(u) = \frac{1}{2} \cdot 2\left(1 - \frac{1-u}{2}\right) = \frac{1+u}{2} \quad 0 \le \frac{1-u}{2} \le 1 \quad$ i.e. $-1 \le u \le 1$

c. $U_3 = Y^2$ $Y = \sqrt{u}$ $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$

$f_{U_3}(u) = \frac{1}{2\sqrt{u}} \cdot 2 \cdot (1 - \sqrt{u}) = \frac{1-\sqrt{u}}{\sqrt{u}}$ $0 \le u \le 1$

6.25 $f(v) = a v^2 e^{-bv^2}$ $v > 0$ $b = m/2kT$

a. $W = mV^2/2$ $V = \sqrt{\frac{2W}{m}}$ $\frac{dv}{dw} = \frac{1}{\sqrt{2mw}}$

$f_W(w) = \frac{1}{\sqrt{2mw}} \cdot a \cdot \frac{2w}{m} \, e^{-b\frac{2w}{m}} = \frac{a\sqrt{2w}}{m\sqrt{m}} e^{-w/kT}$

since $\frac{a\sqrt{2}}{m^{3/2}} = \frac{1}{\Gamma(\frac{3}{2})(kT)^{3/2}}$

$f_W(w) = \frac{1}{\Gamma(\frac{3}{2})(kT)^{3/2}} w^{\frac{1}{2}} e^{-w/kT}$ $w > 0$

$W \sim \text{Gamma}(\frac{3}{2}, kT)$

b. $E(W) = \alpha\beta = \frac{3}{2} kT$

6.26 $f(i) = \frac{1}{2}$ $9 \le i \le 11$

$P = 2I^2$ $I = \sqrt{\frac{P}{2}}$ $\frac{di}{dp} = \frac{1}{2}\left(\frac{P}{2}\right)^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{P}}$

$$f_P(P) = \begin{cases} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{P}} \cdot \frac{1}{2} = \frac{1}{4\sqrt{2P}} & 9 \le \sqrt{\frac{P}{2}} \le 11 \quad \text{i.e. } 162 \le P \le 242 \\ 0 & \text{o.w.} \end{cases}$$

6.60 $F(y) = \frac{y}{\theta}$

a. $$F_{Y_{(n)}}(y) = [F(y)]^n = \begin{cases} 0 & y < 0 \\ \left(\frac{y}{\theta}\right)^n & 0 \le y \le \theta \\ 1 & y > \theta \end{cases}$$

b. $$f_{Y_{(n)}}(y) = F'_{Y_{(n)}}(y) = \begin{cases} \frac{n y^{n-1}}{\theta^n} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$$

c. $E(Y_{(n)}) = \int_0^\theta y \frac{n y^{n-1}}{\theta^n} dy = \frac{n}{\theta^n} \frac{1}{n+1} y^{n+1} \Big|_0^\theta = \frac{n}{n+1}\theta$

$E(Y_{(n)}^2) = \int_0^\theta y^2 \frac{n y^{n-1}}{\theta^n} dy = \frac{n}{\theta^n} \frac{1}{n+2} y^{n+2} \Big|_0^\theta = \frac{n}{n+2}\theta^2$

$V(Y_{(n)}) = E(Y_{(n)}^2) - [E(Y_{(n)})]^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}$

6.61 $P(Y_{(5)} < 10) = F_{Y_{(5)}}(10) = \left(\frac{10}{15}\right)^5 = \left(\frac{2}{3}\right)^5$

6.65 $f(y) = \frac{1}{\beta} e^{-y/\beta}$ $F(y) = 1 - e^{-y/\beta}$ $y > 0$

a. $f_{Y_{(1)}}(y) = n[1 - F(y)]^{n-1} f(y) = n[e^{-y/\beta}]^{n-1} \frac{1}{\beta} e^{-y/\beta}$

$= \frac{n}{\beta} e^{-ny/\beta}$

$Y_{(1)} \sim \text{EXP}\left(\frac{\beta}{n}\right)$ $E(Y_{(1)}) = \frac{\beta}{n}$

b. $f_{Y_{(1)}}(y) = \frac{5}{2} e^{-5y/2}$ $y > 0$

$P(Y_{(1)} \le 3.6) = F_{Y_{(1)}}(3.6) = 1 - e^{-5 \times 3.6/2} = 1 - e^{-9}$

6.68 a. $1 - F(y) = \int_y^\infty e^{-(t-\theta)}\, dt = e^{-(y-\theta)}$

$f_{Y_{(1)}}(y) = n[1 - F(y)]^{n-1} f(y)$

$= n \cdot [e^{-(y-\theta)}]^{n-1} e^{-(y-\theta)}$

$= n e^{-n(y-\theta)}$ $y \ge 0$

b. $E(Y_{(1)}) = \int_0^\infty y\, n\, e^{-n(y-\theta)}\, dy$

$= \int_0^\infty (y - \theta + \theta)\, n\, e^{-n(y-\theta)}\, dy$

$= \int_0^\infty (y-\theta)\, n\, e^{-n(y-\theta)}\, d(y-\theta) + \int_0^\infty \theta n\, e^{-n(y-\theta)}\, dy$

$= n\, \Gamma(2) (\frac{1}{n})^2 + n\theta \frac{1}{n}$

$= \frac{1}{n} + \theta$