5-113 a. $f(y_1, y_2) = \begin{cases} 6(1-y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & o.w. \end{cases}$

$f(y_2) = \int_0^{y_2} 6(1-y_2)\, dy_1 = 6(1-y_2)\, y_2$

$f(y_1 | y_2) = \dfrac{f(y_1, y_2)}{f(y_2)} = \dfrac{1}{y_2} \qquad 0 \le y_1 \le y_2 \le 1$

$E(Y_1 | Y_2 = y_2) = \int_0^{y_2} y_1 \frac{1}{y_2}\, dy_1 = \frac{y_1^2}{2y_2} \Big|_0^{y_2} = \frac{y_2}{2}$

b. $E(Y_1) = E[E(Y_1 | Y_2 = y_2)] = \int_0^1 \frac{y_2}{2}\, 6(1-y_2)\, y_2\, dy_2 = \left[y_2^3 - \frac{3}{4} y_2^4\right]_0^1 = \frac{1}{4}$

5-115 $Y|p \sim Bin(3, p) \Rightarrow E(Y|p) = 3p. \quad V(Y|p) = 3p(1-p)$

$p \sim unif(0, 1) \Rightarrow E(p) = \frac{1}{2} \quad V(p) = \frac{1}{12}$

a. $E(Y) = E[E(Y|p)] = E(3p) = 3E(p) = \frac{3}{2}$

b. $V(Y) = E[V(Y|p)] + V[E(Y|p)]$

$= E[3p(1-p)] + V(3p)$

$= 3E(p) - 3E(p^2) + 9V(p)$

$= 3 \times \frac{1}{2} - 3(V(p) + [E(p)]^2) + 9 \times \frac{1}{12}$

$= 3 \times \frac{1}{2} - 3(\frac{1}{12} + \frac{1}{4}) + 9 \times \frac{1}{12} = \frac{5}{4}$

5-116 $Y|\lambda \sim POI(\lambda) \Rightarrow E(Y|\lambda) = \lambda \quad V(Y|\lambda) = \lambda \qquad \lambda \sim EXP(1) \Rightarrow E(\lambda) = 1 \quad V(\lambda) = 1$

a. $E(Y) = E[E(Y|\lambda)] = E(\lambda) = 1$

b. $V(Y) = E[V(Y|\lambda)] + V[E(Y|\lambda)]$

$= E(\lambda) + V(\lambda)$

$= 1 + 1 = 2$

c. $P(|Y-1| \ge k \cdot \sqrt{2}) \le \frac{1}{k^2}$

$P(|Y-1| \ge 8) = P(|Y-1| \ge 5.567\sigma) \le 0.03125$

$P(Y \ge 9) = P(Y-1 \ge 5.567\sigma) < P(|Y-1| \ge 8) \le 0.03125$

It is unlikely that Y exceeds 9.

5-118 $Y|\lambda \sim POI(\lambda) \Rightarrow E(Y|\lambda) = \lambda \quad V(Y|\lambda) = \lambda$

$\lambda \sim Gamma(\alpha, \beta) \Rightarrow E(\lambda) = \alpha\beta \quad V(\lambda) = \alpha\beta^2$

a. $E(Y) = E[E(Y|\lambda)] = E(\lambda) = \alpha\beta$

b. $V(Y) = E[V(Y|\lambda)] + V[E(Y|\lambda)]$

$= E(\lambda) + V(\lambda)$

$= \alpha\beta + \alpha\beta^2$

$\sigma = \sqrt{\alpha\beta + \alpha\beta^2}$

Example 5.25

$$\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \sim MVN3\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & -0.4 & 1/2 \\ -0.4 & 3 & 2 \\ \frac{1}{2} & 2 & 5 \end{bmatrix} \right)$$

$$\begin{bmatrix} U \\ W \end{bmatrix} \sim MVN2\left(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.4 & \frac{1}{2} \\ -0.4 & 3 & 2 \\ \frac{1}{2} & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

$$= MVN2\left(\begin{bmatrix} -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 12.6 & 2.5 \\ 2.5 & 9.6 \end{bmatrix} \right)$$

$E(U) = -4 \qquad V(U) = 12.6 \qquad cov(U, W) = 2.5$

$E(W) = 5 \qquad V(W) = 9.6$

Multivariate Normal Problems

1. $$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim MVN3\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 10 \end{bmatrix} \right)$$

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sim MVN2\left(\begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

$$= MVN_2\left(\begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}16 & 1\\1 & 32\end{bmatrix}\right)$$

2. $$\begin{bmatrix}W_1\\W_2\\W_3\end{bmatrix} = \begin{bmatrix}1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\begin{bmatrix}X_1\\X_2\\Y_1\\Y_2\\Y_3\end{bmatrix}$$

$$\sim MVN_3\left(\begin{bmatrix}1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\begin{bmatrix}\mu\\\mu\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\end{bmatrix}\begin{bmatrix}\sigma^2&0&0&0&0\\0&\sigma^2&0&0&0\\0&0&\eta^2&0&0\\0&0&0&\eta^2&0\\0&0&0&0&\eta^2\end{bmatrix}\begin{bmatrix}1&1&0\\0&0&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$$

$$= MVN_3\left(\begin{bmatrix}\mu\\\mu\\\mu\end{bmatrix}, \begin{bmatrix}\sigma^2+\eta^2 & \sigma^2 & 0\\\sigma^2 & \sigma^2+\eta^2 & 0\\0 & 0 & \sigma^2+\eta^2\end{bmatrix}\right)$$

3. a. $$\begin{bmatrix}D_1\\D_2\\D_3\end{bmatrix} \sim MVN_3\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$$

$$\begin{bmatrix}L_1\\L_2\\L_3\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}D_1\\D_2\\D_3\end{bmatrix} + \begin{bmatrix}\mu\\\mu\\\mu\end{bmatrix}$$

$$\sim MVN_3\left(\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}\mu\\\mu\\\mu\end{bmatrix}, \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\right)$$

$$= MVN_3\left(\begin{bmatrix}\mu\\\mu\\\mu\end{bmatrix}, \begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}\right)$$

b. $\rho_{1,3} = \dfrac{Cov(L_1, L_3)}{\sqrt{V(L_1) \cdot V(L_3)}} = \dfrac{1}{\sqrt{1 \cdot 3}} = \dfrac{1}{\sqrt{3}}$

c. $L_3 \sim N(\mu, 3)$

$L_3 - \mu \sim N(0, 3)$