5.40 No. $P(Y_1=0, Y_2=-1) = \frac{1}{8} \neq \frac{1}{8}\times\frac{1}{8} = P(Y_1=0)\,P(Y_2=-1)$

5.46 Y_1 and Y_2 are not independent because when $f(y_1, y_2)=1$, $2y_2 \le y_1$, which means the range of y_1 values depends on y_2.

5.51 Y_1 and Y_2 are not independent because when $f(y_1, y_2) = e^{-y_1}$, $0 \le y_2 \le y_1 \le \infty$, which means the range of y_1 values depends on y_2

5.52 Y_1 and Y_2 are not independent because $f(y_1, y_2)$ can not be written as $g(y_1)\cdot h(y_2)$

5.56 $f(y_1) = \begin{cases} 1 & 0<y_1<1 \\ 0 & \text{o.w} \end{cases}$ $\quad f(y_2) = \begin{cases} 1 & 0<y_2<1 \\ 0 & \text{o.w} \end{cases}$

$f(x_1, y_1) = f(x_1)\cdot f(y_1) = \begin{cases} 1 & 0<y_1<1,\ 0<y_2<1 \\ 0 & \text{o.w} \end{cases}$

$P(Y_1<2Y_2,\ Y_1<3Y_2) = P(Y_1<2Y_2) = \int_0^{\frac{1}{2}}\int_0^{2y_2} 1\,dy_1\,dy_2 + \int_{\frac{1}{2}}^{1} 1\,dy_2$

$= y_2^2\Big|_0^{\frac{1}{2}} + y_2\Big|_{\frac{1}{2}}^{1} = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$

$P(Y_1<3Y_2) = \int_0^{\frac{1}{3}}\int_0^{3y_2} 1\,dy_1\,dy_2 + \int_{\frac{1}{3}}^{1} 1\,dy_2$

$= \frac{3}{2}y_2^2\Big|_0^{\frac{1}{3}} + y_2\Big|_{\frac{1}{3}}^{1} = \frac{1}{6}+\frac{2}{3} = \frac{5}{6}$

$P(Y_1<2Y_2 \mid Y_1<3Y_2) = \dfrac{P(Y_1<2Y_2,\ Y_1<3Y_2)}{P(Y_1<3Y_2)} = \dfrac{\frac{3}{4}}{\frac{5}{6}} = \dfrac{9}{10}$

5.59 a. Y_1 and Y_2 are independent

$f(y_1, y_2) = f(y_1)\cdot f(y_2) = \begin{cases} 1/9\ e^{-(y_1+y_2)/3} & y_1>0,\ y_2>0 \\ 0 & \text{o.w.} \end{cases}$

b. $P(Y_1+Y_2\le 1) = \int_0^1\int_0^{1-y_2} f(y_1, y_2)\,dy_1\,dy_2$

$= \int_0^1\int_0^{1-y_2} \frac{1}{9}e^{-(y_1+y_2)/3}\,dy_1\,dy_2$

$= \int_0^1 e^{-y_2/3}\cdot\left[-\frac{1}{3}e^{-y_1/3}\right]_0^{1-y_2} dy_2$

$= \int_0^1 e^{-y_2/3}\cdot\left(\frac{1}{3}-\frac{1}{3}e^{-(1-y_2)/3}\right) dy_2$

$= \int_0^1 \left(\frac{1}{3}e^{-y_2/3} - \frac{1}{3}e^{-1/3}\right) dy_2$

$= -e^{-y_2/3}\Big|_0^1 - \frac{1}{3}e^{-1/3}\,y_2\Big|_0^1$

$= 1-\frac{4}{3}e^{-1/3}$

5.60 $f(y_1, y_2) = f(y_1)\cdot f(y_2) = \begin{cases} 1 & 0\le y_1\le 1,\ 0\le y_2\le 1 \\ 0 & \text{o.w.} \end{cases}$

$P(Y_2 \le Y_1 \le Y_2 + \frac{1}{4}) = \int_0^{1/4} \int_0^{y_1} 1\, dy_2\, dy_1 + \int_{1/4}^{1} \int_{y_1-1/4}^{y_1} 1\, dy_2\, dy_1$

$= \int_0^{1/4} y_1\, dy_1 + \int_{1/4}^{1} \frac{1}{4}\, dy_1$

$= \frac{1}{2} y_1^2 \Big|_0^{1/4} + \frac{1}{4} y_1 \Big|_{1/4}^{1} = \frac{1}{32} + \frac{1}{4} - \frac{1}{16} = \frac{7}{32}$

5-62 a. $E(Y_1) = np = 2 \times \frac{1}{3} = \frac{2}{3}$

b. $V(Y_1) = np(1-p) = 2 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$

c. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2)$

since $E(Y_2) = E(Y_1) = \frac{2}{3}$

$E(Y_1 - Y_2) = 0$

5-66 a. $f(y_1, y_2) = \begin{cases} 1 & 0 \le y_1 \le 2,\ 0 \le y_2 \le 1,\ 2y_2 \le y_1 \\ 0 & \text{o.w.} \end{cases}$

$f(y_1) = \int_0^{y_1/2} 1\, dy_2 = \frac{y_1}{2} \qquad 0 \le y_1 \le 2$

$f(y_2) = \int_{2y_2}^{2} 1\, dy_1 = 2 - 2y_2 \qquad 0 \le y_2 \le 1$

$E(Y_1) = \int_0^2 y_1 \frac{y_1}{2}\, dy_1 = \frac{y_1^3}{6}\Big|_0^2 = \frac{4}{3}$

$E(Y_2) = \int_0^1 y_2(2-2y_2)\, dy_2 = y_2^2 - \frac{2}{3} y_2^3 \Big|_0^1 = \frac{1}{3}$

b. $E(Y_1^2) = \int_0^2 y_1^2 \frac{y_1}{2}\, dy_1 = \frac{y_1^4}{8}\Big|_0^2 = 2$

$V(Y_1) = E(Y_1^2) - [E(Y_1)]^2 = 2 - \frac{16}{9} = \frac{2}{9}$

$E(Y_2^2) = \int_0^1 y_2^2 (2-2y_2)\, dy_2 = \frac{2}{3} y_2^3 - \frac{1}{2} y_2^4 \Big|_0^1 = \frac{1}{6}$

$V(Y_2) = E(Y_2^2) - [E(Y_2)]^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$

c. $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = \frac{4}{3} - \frac{1}{3} = 1$

d. $V(Y_1 - Y_2) = V(Y_1) + V(Y_2) - 2\,Cov(Y_1, Y_2)$

$Cov(Y_1, Y_2) = E(Y_1, Y_2) - E(Y_1)\cdot E(Y_2)$

$E(Y_1, Y_2) = \int_0^1 \int_{2y_2}^{2} y_1 y_2\, dy_1\, dy_2$

$= \int_0^1 \frac{y_2}{2}(4 - 4y_2^2)\, dy_2 = y_2^2 - \frac{1}{2} y_2^4 \Big|_0^1 = \frac{1}{2}$

Hence $V(Y_1 - Y_2) = \frac{2}{9} + \frac{1}{18} - 2\left(\frac{1}{2} - \frac{4}{3} \times \frac{1}{3}\right)$

$= \frac{1}{6}$

Using Tchebysheff's theorem, with $k = 2$

$P(\mu - 2\sigma \le Y_1 - Y_2 \le \mu + 2\sigma) \ge \frac{3}{4}$

Thus $P(1 - 2 \times \sqrt{\frac{1}{6}} \le Y_1 - Y_2 \le 1 + 2\sqrt{\frac{1}{6}}) \ge \frac{3}{4}$

$\Rightarrow P(0.18 \le Y_1 - Y_2 \le 1.82) \ge \frac{3}{4}$

5-68 $f(y_1, y_2) = \begin{cases} y_1 + y_2 & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0 & \text{o.w.} \end{cases}$

$f(y_1) = \int_0^1 (y_1 + y_2)\, dy_2 = y_1 y_2 + \frac{1}{2} y_2^2 \Big|_0^1 = y_1 + \frac{1}{2} \qquad 0 \le y_1 \le 1$

similarly $f(y_2) = y_2 + \frac{1}{2} \qquad 0 \le y_2 \le 1$

$E(Y_1) = \int_0^1 y_1 (y_1 + \frac{1}{2})\, dy_1 = \frac{1}{3} y_1^3 + \frac{1}{4} y_1^2 \Big|_0^1 = \frac{7}{12}$

similarly $E(Y_2) = \frac{7}{12}$

$E(30Y_1 + 25Y_2) = 30E(Y_1) + 25E(Y_2) = 30 \times \frac{7}{12} + 25 \times \frac{7}{12} = \frac{385}{12} = 32.08$

5-75 $Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$E(Y_1 Y_2) = 0 \cdot 0 \cdot \frac{1}{9} + 0 \cdot 1 \cdot \frac{2}{9} + 0 \cdot 2 \cdot \frac{1}{9} + 1 \cdot 0 \cdot \frac{2}{9} + 1 \cdot 1 \cdot \frac{2}{9}$
$+ 1 \cdot 2 \cdot 0 + 2 \cdot 0 \cdot \frac{1}{9} + 2 \cdot 1 \cdot 0 + 2 \cdot 2 \cdot 0$
$= \frac{2}{9}$

we have calculated in 5-62 that

$E(Y_1) = E(Y_2) = \frac{2}{3}$

$Cov(Y_1, Y_2) = \frac{2}{9} - \frac{2}{3} \times \frac{2}{3} = -\frac{2}{9}$

It is not surprising that $Cov(Y_1, Y_2)$ is negative. It is because when Y_1 increases, Y_2 tends to decrease.

5-83 a. $Cov(Y_1, Y_1) = V(Y_1) = 2$

b. If $Cov(Y_1, Y_2) = 7$ $\rho = 7/\sqrt{2} \cdot \sqrt{8} = 1.75 > 1$

It is not possible

c. If $\rho = 1$ $Cov(Y_1, Y_2) = \sqrt{2} \cdot \sqrt{8} = 4$ It is the largest possible value for $Cov(Y_1, Y_2)$, which implies perfect positive relationship.

d. If $\rho = -1$ $Cov(Y_1, Y_2) = -4$. It is the smallest possible value for $Cov(Y_1, Y_2)$, which implies perfect negative relationship.

5-86 X = dollar spent per week $X = 3Y_1 + 5Y_2$

$E(X) = E(3Y_1 + 5Y_2) = 3E(Y_1) + 5E(Y_2) = 3 \times 40 + 5 \times 65 = 445$

$V(X) = V(3Y_1 + 5Y_2) = 9V(Y_1) + 25V(Y_2)$ (Y_1, Y_2 independent)
$= 9 \times 4 + 25 \times 8 = 236$

5-87 $E(3Y_1 + 4Y_2 - 6Y_3) = 3E(Y_1) + 4E(Y_2) - 6E(Y_3)$
$= 3 \times 2 + 4 \times (-1) - 6 \times 4 = -22$

$V(3Y_1 + 4Y_2 - 6Y_3) = 9V(Y_1) + 16V(Y_2) + 36V(Y_3) + 24\,Cov(Y_1, Y_2)$
$- 36\,Cov(Y_1, Y_3) - 48\,Cov(Y_2, Y_3)$
$= 9 \times 4 + 16 \times 6 + 36 \times 8 + 24 \times 1 - 36 \times (-1) - 48 \times (0) = 480$

5-93. From 5-68, we got $E(Y_1) = E(Y_2) = \frac{7}{12}$

$E(Y_1^2) = \int_0^1 y_1^2 (y_1 + \frac{1}{2})\, dy_1 = \frac{1}{4} y_1^4 + \frac{1}{6} y_1^3 \Big|_0^1 = \frac{5}{12}$

similarly, $E(Y_2^2) = \frac{5}{12}$

$V(Y_1) = E(Y_1^2) - [E(Y_1)]^2 = \frac{5}{12} - (\frac{7}{12})^2 = \frac{11}{144}$

similarly, $V(Y_2) = \frac{11}{144}$

$E(Y_1 Y_2) = \int_0^1 \int_0^1 y_1 y_2 (y_1 + y_2)\, dy_1\, dy_2$

$= \int_0^1 [\frac{1}{3} y_2 y_1^3 + \frac{1}{2} y_2^2 y_1^2]_0^1\, dy_2$

$= \int_0^1 (\frac{1}{3} y_2 + \frac{1}{2} y_2^2)\, dy_2$

$= \frac{1}{6} y_2^2 + \frac{1}{6} y_2^3 \Big|_0^1 = \frac{1}{3}$

$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1) E(Y_2)$

$= \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = -\frac{1}{144}$

$V(30Y_1 + 25Y_2) = 30^2 V(Y_1) + 25^2 V(Y_2) + 1500\, Cov(Y_1, Y_2)$

$= 900 \times \frac{11}{144} + 625 \times \frac{11}{144} + 1500 \times (-\frac{1}{144})$

$= 15275/144 = 106.08$

$\sigma = \sqrt{V(30Y_1 + 25Y_2)} = 10.30$

$E(30Y_1 + 25Y_2) = 32.08$ (by 5-68)

Using Tchebysheff's theorem with $k=2$.

$P(\mu - 2\sigma \le 30Y_1 + 25Y_2 \le \mu + 2\sigma) \ge \frac{3}{4}$

The interval is $32.08 \pm 2 \times 10.30$. i.e. $(11.48, 52.68)$