Stat 447 Final Exam

May 6, 2002 Prof. Vardeman

1. An example in Moore's *Basic Practice of Statistics* concerns the deterioration of synthetic fiber when buried in a landfill. A total of 10 strips of a polyester fabric were buried in well-drained soil. $n_1 = 5$ strips were then dug up and strength tested after 2 weeks, and the other $n_2 = 5$ were dug up and strength tested after 16 weeks. The mean and standard deviation of the first 5 measured strengths were respectively 123.8 lbs and 4.6 lbs. The mean and standard deviation of the last 5 measured strengths were respectively 116.4 lbs and 16.1 lbs.

a) Give 95% two-sided confidence limits for the mean strength of strips of this polyester buried for 16 weeks based on a normal distribution assumption. (Plug in correctly, but there is no need to simplify.)

Use $\bar{x} \pm t\frac{s}{\sqrt{n}}$ This is.

$$116.4 \pm 2.776\frac{16.1}{\sqrt{5}}$$

b) There is some hint in the data summaries above that the longer polyester is buried, the less consistent is its breaking strength. How strong do you judge this evidence to be? Give some quantitative analysis to support your answer.

Note that $\frac{s_2^2}{s_1^2} = \left(\frac{16.1}{4.6}\right)^2 = 12.25$ which falls between the upper .025 point and the upper .01 pt of the $F_{4,4}$ dsn. A p-value for $H_0: \sigma_2^2 = \sigma_1^2$ vs $H_a: \sigma_2^2 > \sigma_1^2$ is thus between .025 and .01 and even with $H_a: \sigma_2^2 \neq \sigma_1^2$ the p-value is between $2(.025) = .05$ and $2(.01) = .02$. This is reasonably strong evidence that σ is not constant

Henceforth ignore any misgivings about change in consistency of breaking strength raised in b).

c) Can the null hypothesis of equality of mean strengths after 2 and after 16 weeks be rejected in favor of a decrease in mean strength with $\alpha = .05$? Show appropriate work to support your answer.

Assuming it is permissible to act as if $\sigma_1 = \sigma_2$ we estimate σ with

$$s_p = \sqrt{\frac{4(4.6)^2 + 4(16.1)^2}{8}} = 11.84 \quad \text{so} \quad t = \frac{123.8 - 116.4 - 0}{11.84\sqrt{\frac{1}{5}+\frac{1}{5}}}$$

$$= .98$$

The rejection region is $T >$ upper .05 pt of t_8 dsn i.e. $T > 1.860$. Since $.98 < 1.860$ we do not reject H_0!

2. A data set in Moore's *Basic Practice of Statistics* gives heating degree days x_i and natural gas consumption y_i (in 100 cu ft) for $n=16$ months for a particular household. These produced $b_0 = 1.0892$, $b_1 = .188999$ and $\sqrt{SSE/14} = .3389$. The $n=16$ months had $\bar{x} = 22.3$ and $\sum_{i=1}^{16}(x_i - \bar{x})^2 = 4,719$.

a) Give 90% two-sided confidence limits for the standard deviation of gas consumption for any fixed number of heating degree days based on the fact that SSE/σ^2 has a χ^2 distribution under the simple linear regression model. (Plug in, but you need not simplify.)

By analogy with the one-sample interval for σ, we can use limits $\left(\sqrt{\frac{SSE}{U}}, \sqrt{\frac{SSE}{L}}\right)$. Here that is

$$\left(\sqrt{\frac{(.3389)^2(14)}{23.6848}}, \sqrt{\frac{(.3389)^2(14)}{6.57063}}\right)$$

i.e. $(.2606, .4947)$

b) Give 90% two-sided confidence limits for the increase in mean gas consumption per 1 degree day increase in x (that is, for β_1). (Plug in, but you need not simplify.)

Use $b_1 \pm t\dfrac{\sqrt{SSE/(n-2)}}{\sqrt{\sum(x_i-\bar{x})^2}}$ Here that is

$$.188999 \pm 1.761\frac{.3389}{\sqrt{4,719}}$$

i.e. $.188999 \pm .0087$

c) Give 90% two-sided prediction limits for the next gas consumption for this household in a month in which $x = 30$. (Pug in, but you need not simplify.)

Use $(b_0 + b_1 x_{n+1}) \pm t\sqrt{\frac{SSE}{n-2}}\sqrt{1+\frac{1}{n}+\frac{(x_{n+1}-\bar{x})^2}{\sum(x_i-\bar{x})^2}}$

Here that is

$$\left[1.0892 + (.188999)(30)\right] \pm 1.761(.3389)\sqrt{1+\frac{1}{16}+\frac{(30-22.3)^2}{4,719}}$$

i.e. $6.76 \pm .62$

3. An example used in class is the estimation of a parameter θ based on observations $X_1, X_2, \ldots, X_n$ modeled as iid Uniform $(0,\theta)$. We determined that $\hat{\theta} = \max_{i=1,\ldots,n} X_i$ is the maximum likelihood estimator of θ and that $\tilde{\theta} = 2\bar{X}$ is the method of moments estimator of θ.

a) Show that $E\hat{\theta} = \left(\frac{n}{n+1}\right)\theta$ (so that $\hat{\theta}$ is a biased estimator of θ).

$$E\hat{\theta} = \int_0^\theta x n \left(\frac{x}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right)dx = \frac{n}{\theta^n}\int_0^\theta x^n\,dx$$

see pdf on page 318

$$= \frac{n}{\theta^n}\left.\frac{x^{n+1}}{n+1}\right|_0^\theta = \frac{n}{n+1}\theta$$

You may henceforth assume without proof that $V(\hat{\theta}) = \frac{\theta^2}{(n+2)(n+1)^2}$.

b) It is obvious from the result of part a) that $\hat{\theta}^* = \left(\frac{n+1}{n}\right)\hat{\theta}$ is an unbiased estimator of θ closely related to $\hat{\theta}$. For what sample sizes do you prefer $\hat{\theta}^*$ to the (unbiased) method of moments estimator $\tilde{\theta}$? (Consider variances.)

$$V(\tilde{\theta}) = 2^2 V(\bar{X}) = 4\,\frac{\theta^2}{12}\left(\frac{1}{n}\right) = \frac{\theta^2}{3n}$$

$$V(\hat{\theta}^*) = \left(\frac{n+1}{n}\right)^2 \frac{\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{(n+2)n^2}$$

For $n=1$ these variances are the same (the estimators are the same!). For $n>1$ $V(\hat{\theta}^*)$ is smaller than $V(\tilde{\theta})$.

c) Suppose $n = 100$. Which estimator of θ do you prefer, the MLE $\hat{\theta}$ or its unbiased multiple $\hat{\theta}^*$ (that has a larger variance)? Make your comparison on the basis of mean squared error.

MSE = Variance + Bias^2

For $\hat{\theta}^*$ This is $V(\hat{\theta}^*) = \frac{\theta^2}{(102)(100)^2}$

For $\hat{\theta}$ this is $MSE = \frac{\theta^2}{(102)(101)^2} + \left(\theta - \frac{100}{101}\theta\right)^2$

$$= \theta^2\left[\frac{1}{102(101)^2} + \frac{1}{(101)^2}\right] = \frac{103\,\theta^2}{(102)(101)^2}$$

The first is substantially smaller than the 2nd – The unbiased multiple is better.

4. Suppose that observations $X_1, X_2, \ldots, X_n$ are modeled as iid $N(\mu, \sigma^2)$.

a) Find a sample size n so that with probability .90, the sample mean is within $.1\sigma$ of μ.

$\bar{X}$ is normal with mean μ and variance $\frac{\sigma^2}{n}$. We want z-value for $\mu + .1\sigma$ to be 1.645.

$$\frac{\mu + .1\sigma - \mu}{\sigma/\sqrt{n}} = 1.645$$

implies $\quad .1\sqrt{n} = 1.645 \quad$ i.e. $\quad n = 271$

b) Suppose that X_{n+1} is a single additional observation from this normal distribution. What is the distribution of $\bar{X} - X_{n+1}$? (The sample mean $\bar{X}$ is based on the original n observations.) Specify the distribution completely (give appropriate parameter values).

$\bar{X} - X_{n+1}$ is normal with mean

$$E(\bar{X} - X_{n+1}) = E\bar{X} - EX_{n+1} = \mu - \mu = 0$$

and variance

$$\begin{aligned} V(\bar{X} - X_{n+1}) &= 1^2 V(\bar{X}) + (-1)^2 V(X_{n+1}) + 2(1)(-1)\text{Cov}(\bar{X}, X_{n+1}) \\ &= \frac{\sigma^2}{n} + \sigma^2 + 0 \\ &= \sigma^2\left(1 + \frac{1}{n}\right) \end{aligned}$$

c) The sample standard deviation based on the original n observations, s, is independent of $\bar{X} - X_{n+1}$. It is possible to build a variable from these two variables that has a t_{n-1} distribution. Do this (give the formula for such a t_{n-1} variable).

$\dfrac{\bar{X} - X_{n+1}}{\sigma\sqrt{1 + \frac{1}{n}}}$ is standard normal, and the usual gyrations show that $\quad \dfrac{\bar{X} - X_{n+1}}{s\sqrt{1 + \frac{1}{n}}} \sim t_{n-1}$

5. The decay of a radioactive material is such that the number of emitted particles detected by a certain Geiger counter in 1 second is a Poisson random variable with mean λ (and the number detected in k seconds is Poisson with mean $k\lambda$).

a) Suppose that I plan to make a count X of particles detected in 1 second, intending to test $H_0: \lambda = 1$ versus $H_a: \lambda > 1$. I determine to reject H_0 if $X \geq 4$. Use the book's table of Poisson probabilities and evaluate the type I error probability (α) I am using.

$$\alpha = P_{\lambda=1}[X \geq 4] = 1 - P_{\lambda=1}[X \leq 3] = 1 - .981 = .019$$

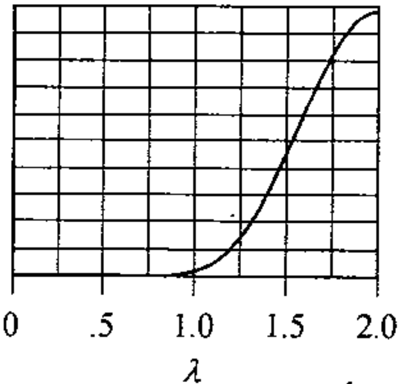
b) In the context of part a), suppose that in fact $\lambda = 2$. What is the type II error probability (β)?

$$\beta = P_{\lambda=2}[X < 4] = P_{\lambda=2}[X \leq 3] = .857$$

c) Suppose that "*a priori*" I model λ as Uniform $(0,2)$. What is my "prior" probability that $\lambda > 1.5$?

$$P[\lambda > 1.5] = \int_{1.5}^{2} \frac{1}{2}\,dx = \frac{1}{2}x\Big|_{1.5}^{2} = \frac{1}{4}$$

d) If I use the "prior" distribution of part c) and in 10 seconds detect 20 particles, the posterior distribution of λ (that is NOT of a standard type) has a pdf proportional to $\exp(-10\lambda)\lambda^{20}$ on the interval $(0,2)$. This is pictured below. Approximate the "posterior" probability that $\lambda > 1.5$.



"counting boxes" on the graph, the fraction of the area under the curve that is to the right of 1.5 is about

$$\frac{15.9}{20.4} \simeq .8$$

(the "exact" value is actually .812)

6. Consider the continuous distribution with pdf

$$f(x\mid p)=(1-p)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)+p\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-1)^2}{2}\right)$$

for $0\le p\le 1$. This pdf is appropriate if an observable is $\mathrm{N}(1,1)$ with probability p and is $\mathrm{N}(0,1)$ with probability $1-p$. The distribution has mean p and variance $1+p-p^2$.

This problem concerns inference for p based on observations $X_1,X_2,\ldots,X_n$ that are iid according to $f(x\mid p)$. Attached is a printout made using MATHCAD for the analysis of a particular set of $n=25$ data values. The function l is the loglikelihood, the function *lprime* is the score function, and the function *lprimeprime* is the derivative of the score function. The 25 observations have $\bar{x}=.070$ and $s=1.094$ and the printout shows $lprime(.104)=0$ and $lprimeprime(.104)=-29.106$. GIVE NUMERICAL ANSWERS TO THE QUESTIONS BELOW BASED ON THIS SAMPLE.

a) Give both the maximum likelihood estimate $\hat{p}$ and the method of moments estimate $\tilde{p}$ for p.

$EX=p \Rightarrow$ method of moments gives $\tilde{p}=\bar{x}$

$\hat{p}=$ .104 $\qquad\qquad$ $\tilde{p}=$ .070

b) Use the maximum likelihood estimate and find approximate 90% two-sided confidence limits for p based on it.

Use $\hat{\theta}\pm z\sqrt{\frac{-1}{\ell''(\hat{\theta})}}$ Here that is

$.104\pm 1.645\sqrt{\frac{1}{29.106}}$ i.e. $.104\pm .305$

c) Make 90% two-sided confidence limits for p based on $\bar{x}$ and s.

Since $p=\mu=EX$ we can use the large sample confidence limits for a mean

$\bar{x}\pm 1.645\frac{s}{\sqrt{n}}$ i.e. $.070\pm 1.645\frac{1.094}{\sqrt{25}}$

i.e. $.070\pm .360$

d) Give an approximate observed value for the likelihood ratio statistic for testing $\mathrm{H}_0: p=.9$.

$\lambda=\frac{\max_p L(p)}{\max_{p=.9} L(p)}$ So $\ln\lambda=\ln\max_p L(p)-\ln L(.9)$

$=\max_p \ln L(p)-\ln L(.9)$

$=\max_p \ell(p)-\ell(.9)$

$\approx -37.206-(-45.058)=7.852$

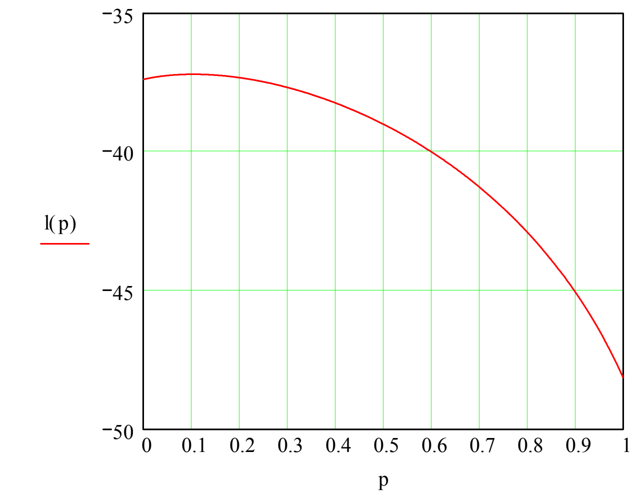
And $\lambda\approx e^{7.852}=2,571$

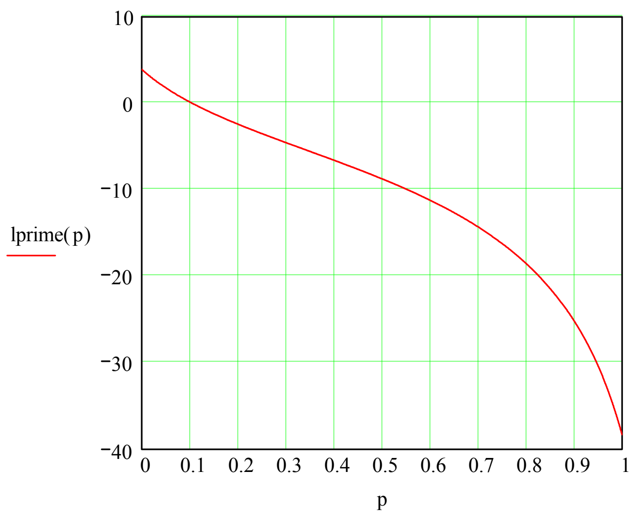
$$x := \begin{bmatrix} -.5 \\ -.14 \\ -.82 \\ .86 \\ 2.05 \\ 2.29 \\ .97 \\ -.37 \\ -.07 \\ -.71 \\ -1.39 \\ .63 \\ 1.33 \\ .05 \\ -1.65 \\ -1.33 \\ .52 \\ .1 \\ .57 \\ -.71 \\ 1.32 \\ 1.03 \\ -1.08 \\ .33 \\ -1.53 \end{bmatrix}$$

$$l(p) := \sum_{i=0}^{24} \ln\left[(1-p)\cdot \mathrm{dnorm}\left(x_i, 0, 1\right) + p\cdot \mathrm{dnorm}\left(x_i, 1, 1\right)\right]$$

$$\mathrm{lprime}(p) := \frac{d}{dp} l(p)$$

$$\mathrm{lprimeprime}(p) := \frac{d^2}{dp^2} l(p)$$





$p := .1$

$a := \mathrm{root}(\mathrm{lprime}(p), p)$

$a = 0.104$

$l(a) = -37.206 \qquad l(.9) = -45.058$

$\mathrm{lprimeprime}(a) = -29.106$

$\mathrm{mean}(x) = 0.07$

$\mathrm{stdev}(x) = 1.072$ (this is the "n" divisor standard deviation, not the usual one)