

1. Diseases A and B in sheep have similar symptoms, but never afflict the same animal. B is more rare and is harder to diagnose. 1 in 1000 sheep have disease A, and 1 in 10,000 have disease B. In routine exams, veterinarians fail to diagnose A in 50% of diseased animals and fail to diagnose B in 80% of diseased animals. Animals without diseases A and B are never diagnosed as having one of the diseases. No animal with A (B) is ever diagnosed as having B (A).

For parts (a) through (c) consider a single sheep given a routine exam by a veterinarian.

- 7 pts a) What is P [sheep has disease A or B]? "has A" and "has B" are mutually exclusive

$$P[\text{sheep has A or B}] = .001 + .0001 \\ = .0011$$

- 7 pts b) What is P [sheep is diagnosed with either A or B]? again because they are mutually exclusive events

$$P[\text{diagnosed with A or B}] = P[\text{diagnosed with A}] + P[\text{diagnosed with B}] \\ = P[\text{diagnosed with A} | \text{has A}] P[\text{has A}] \\ + P[\text{diagnosed with B} | \text{has B}] P[\text{has B}] \\ = (.5)(.001) + (.2)(.0001) = .00052$$

- 7 pts c) What is P [sheep is free of both A & B | sheep is diagnosed as free of A and B]?

$$P[\text{free of A+B} | \text{diagnosed free of A+B}] = \frac{P[\text{free and diagnosed free}]}{P[\text{diagnosed free}]} \\ = \frac{.9989}{.9989 + (.5)(.001) + (.8)(.0001)} \\ = .9994$$

- 10 pts d) Suppose that one samples 100 sheep at random with replacement (from all sheep in the world) and subjects them to a routine exam by a veterinarian. With

X = the number of sheep diagnosed with A or B

use your answer to (b) (or the number .00064 if you weren't able to do (b)) and find EX and $P[X=0]$.

$$X \sim \text{Binomial}(100, .00052)$$

$$EX = np = .052 \quad P[X=0] = \binom{100}{0} (.00052)^0 \times (.99948)^{100} = .9493$$

$$EX = \underline{.052}$$

$$P[X=0] = \underline{.9493}$$

- 5 pts e) In contrast to part (d), suppose that the 100 sheep in the Iowa flock of Farmer Jones are examined by a veterinarian. With X as before, why would it now be unwise to use the calculations of (d) to describe X ?

I probably shouldn't model the "trials" here as independent. Sheep in the same flock are more likely to be alike than ones from different flocks, either by genetics or infection.

2. Iowa vehicle license plates have identifiers of the form XXX-YYY where the X's are digits 0-9 and the Y's are letters A-Z. (There are 26 letters in the alphabet.)

- 7 pts a) How many different identifiers of this type are possible?

$$10^3 \times (26)^3 = 17,576,000$$

- 7 pts b) I remember that my plate has the digits 2-5-7 in some order and letters A-D-Y in some order but beyond this recall nothing about the identifier. Loving things random, I "randomly" order the digits and then the letters. What is the probability that doing so, I recreate my license plate number?

$$P[\text{correct plate number}] \stackrel{\text{independence}}{=} P[\text{right order for digits}] P[\text{right order for letters}] = \frac{1}{P_3^3} \cdot \frac{1}{P_3^3} = \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) = \frac{1}{36}$$

3. At one stage of the manufacture of certain fiber-optic cable, serious flaws occur at a rate of .2 flaws/mile of cable.

- 8pts a) Find a sensible value for the probability that 20 miles of this cable has less than 2 serious flaws.

$X = \text{total flaws} - \text{Model } X \sim \text{Poisson}((.2)(20)) \text{ i.e.}$
 Poisson with mean 4

$$P[X < 2] = P(0) + P(1) = \frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} = 5e^{-4} = .0916$$

- 10pts b) Suppose that one buys 2000 miles of this cable and $X =$ the total number of serious flaws. "Exact" calculation of $P[X \leq 380]$ is not pleasant "by hand." However, the (intrinsically discrete) distribution of X can be approximated with a (continuous) normal distribution. Find an appropriate mean and standard deviation for X (μ and σ), treat X as if it were normal, and approximate $P[X \leq 380]$.

I'd like to model $X \sim \text{Poisson}((.2)(2000))$ i.e. Poisson with mean 400, but don't have tables or access to a program for calculating $P[X \leq 380]$ - For this den $EX = 400 = \mu$ and $\sqrt{\text{Var } X} = \sqrt{400} = 20 = \sigma$ - Approximate the desired probability with a normal probability for $\mu = 400$ and $\sigma = 20$

$$P[X \leq 380] = P\left[\frac{X-400}{20} \leq -1\right] \approx P\left[Z \leq -1\right] = .1587$$

 std normal

4. Suppose that U is Uniform(0,1). (Note then that $EU = \frac{1}{2}$, $EU^2 = \frac{1}{3}$, $EU^3 = \frac{1}{4}$ and $EU^4 = \frac{1}{5}$.)

- 10pts a) $W = 3U^2 + 5$ is a random variable with a mean EW and variance $V(W)$. Find these.

$$EW = E(3U^2 + 5) = 3EU^2 + 5 = 3\left(\frac{1}{3}\right) + 5 = 6$$

$$V(W) = V(3U^2 + 5) = 9V(U^2) = 9[E U^4 - (EU^2)^2]$$

$$= 9\left[\frac{1}{5} - \frac{1}{9}\right] = \frac{4}{5}$$

$$EW = \underline{6}$$

$$V(W) = \underline{\frac{4}{5}}$$

- 6pts b) It is even possible to work out probabilities for W defined in part (a). For example, find $P[W \leq 6]$.
(Hint: What is this probability in terms of U ?)

$$\begin{aligned} P[W \leq 6] &= P[3U^2 + 5 \leq 6] = P[3U^2 \leq 1] \\ &= P[U^2 \leq \frac{1}{3}] \\ &= P[U \leq \frac{1}{\sqrt{3}}] \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

5. Suppose that X is a continuous random variable with pdf

$$f(x) = \begin{cases} 1/4 & \text{if } x \in (0,1) \\ 3/4 & \text{if } x \in (1,2) \\ 0 & \text{otherwise} \end{cases}$$

- 6pts a) Find $P[X \leq 1.2]$.

$$\begin{aligned} P[X \leq 1.2] &= \int_{-\infty}^{1.2} f(x) dx = \int_0^1 \frac{1}{4} dx + \int_1^{1.2} \frac{3}{4} dx \\ &= \frac{1}{4} + (.2) \left(\frac{3}{4} \right) = \frac{10}{40} + \frac{6}{40} = .4 \end{aligned}$$

- 10pts b) Find the mean and variance of X .

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} xf(x) dx = \frac{1}{4} \int_0^1 x dx + \frac{3}{4} \int_1^2 x dx \\ &= \frac{1}{4} \left. \frac{x^2}{2} \right|_0^1 + \frac{3}{4} \left. \frac{x^2}{2} \right|_1^2 = \frac{1}{8} + \frac{9}{8} = \frac{5}{4} \end{aligned}$$

$$\begin{aligned} EX^2 &= \frac{1}{4} \int_0^1 x^2 dx + \frac{3}{4} \int_1^2 x^2 dx = \frac{1}{4} \left. \frac{x^3}{3} \right|_0^1 + \frac{3}{4} \left. \frac{x^3}{3} \right|_1^2 \\ &= \frac{1}{12} + \frac{21}{12} = \frac{22}{12} \end{aligned}$$

$$V(X) = \frac{22}{12} - \left(\frac{5}{4} \right)^2 = \frac{13}{48}$$

$$EX = \underline{\underline{\frac{5}{4}}}$$

$$V(X) = \underline{\underline{\frac{13}{48}}}$$