401B Assignment Solution 4

1. Section 3 Exercises (all) Section 6.3 of Vardeman and Jobe (page 385).

2. Section 4, Exercises (all) Section 6.4 of Vardeman and Jobe (page 399).

3. Section 5, Exercises (all) Section 6.5 of Vardeman and Jobe (page 413).

4. Section 6, Exercises 2 and 3 Section 6.6 of Vardeman and Jobe (pages 426 and 427).

Solutions for Section 6.3 all

1. The normal distribution is bell-shaped and symmetric, with no outliers. The confidence interval methods depend on this regularity. If the distribution is skewed or prone to outliers, the normal-theory methods will not properly take this into account. The result is an interval whose real confidence level is lower than the nominal value associated with it. For example, if the data are skewed to the right (long right tail), a 90% normal-theory confidence interval for the mean will tend to underestimate the mean, and so the method will produce intervals that contain the mean less than 90% of the time.

2. (a) It is required that the top bolt torques for each piece are independent and approximately normally distributed. The normal probability plot suggests the torque values for the top bolt come from a normal distribution.

(b) Ho: $\mu = 100$ vs. Ha: $\mu \neq 100$ \[ t = \frac{(\bar{x} - 100)}{(s/\sqrt{n})} = \frac{(111 - 100)}{(9.6732/\sqrt{15})} = 4.4. \] $p$-value = 2 $P \{ t_{14} > 4.4 \} = .001$. Conclude Ha: $\mu \neq 100$.

(c) $\bar{x} \pm ts/\sqrt{n}$ becomes $111 \pm (2.624)(9.6732)/\sqrt{15}$ or $111 \pm 6.553$. The interval [104.45, 117.55] is a 98% confidence interval for the mean torque for the top bolt.

(d) Since the data are paired, one should take the difference for each pair and analyze the differences. This is a small sample (small number of pairs), so the differences need to be iid normal to use the methods in Section 6.3. One way to check this assumption is to make a normal plot of the differences. (I have taken the differences as Top−Bottom.)
Given the number of ties in the data, this plot is fairly linear, indicating that the differences are roughly bell-shaped. Other than the discrete (chunky) nature of the data, there is no evidence against the assumption of a normal distribution for the differences.

(e) 1. \( H_0: \mu_d = 0 \).
2. \( H_A: \mu_d < 0 \).
3. The test statistic is given by equation (6-26), with \( \delta = 0 \). The reference distribution is the \( t_{14} \) distribution. Observed values of \( T \) far below zero will be considered as evidence against \( H_0 \).
4. The sample gives

\[ t = -2.10. \]

5. The observed level of significance is

\[ P(\text{a } t_{14} \text{ random variable } < -2.10) = P(\text{a } t_{14} \text{ random variable } > 2.10) \]

which is between \(.025\) and \(.05\), according to Table B-4. This is fairly strong evidence that there is a mean increase in required torques as one moves from the top to the bottom bolts.

(f) Use equation (6-25). For 98% confidence, the appropriate \( t \) is \( t = Q_{14}(.99) = 2.624 \), from Table B-4.

\[ -6.0 \pm 2.624 \left( \frac{11.0518}{\sqrt{15}} \right) = -6.0 \pm 7.4878 \]
\[ = [-13.49, 1.49]. \]

3. (a) Use equation (6-22). For 90% confidence, the appropriate \( z \) is 1.645. The interval is

\[ .0004 \pm 1.645 \left( \frac{.01159873}{\sqrt{50}} \right) = .0004 \pm .002698308 \]
\[ = [-.0023, .0031] \text{ mm}. \]

(b) 1. \( H_0: \mu_d = 0 \).
2. \( H_A: \mu_d \neq 0 \).
3. The test statistic is given by equation (6-24), with \( \# = 0 \). The reference distribution is the standard normal distribution. Observed values of \( Z \) far above or below zero will be considered as evidence against \( H_0 \).
4. The sample gives

\[ z = .24. \]

5. The observed level of significance is

\[ 2P(\text{a standard normal random variable } > .24) = 2P(\text{a standard normal random variable } < -.24) \]
which is equal to $2(.4052) = .8104$, according to Table D-3. There is no evidence of a systematic difference in the readings produced by the two calipers.

(c) The confidence interval in part (a) contains zero; in fact, zero is near the middle of the interval. This means that zero is a very plausible value for the mean difference—there is no evidence that the mean is not equal to zero. This is reflected by the large $p$-value in part (b).

4. (a) The data within each sample must be iid normal, and the two distributions must have the same variance $\sigma^2$. One way to check these assumptions is to normal plot each data set on the same axes (see Figure 6-15).

For such small sample sizes, it is difficult to verify the assumptions. The plots are roughly linear with no outliers, indicating that the normal part of the assumption may be reasonable. The slopes are similar, indicating that the common variance assumption may be reasonable.

(b) Label the Treaded data Sample 1 and the Smooth data Sample 2.

1. $H_0: \mu_1 - \mu_2 = 0$.
2. $H_a: \mu_1 - \mu_2 \neq 0$.
3. The test statistic is given by equation (6-36) with # = 0, and the reference distribution is the $t_{10}$ distribution. Observed values of $T$ far above or below zero will be considered as evidence against $H_0$.
4. The sample gives

$$t = 2.49$$

5. The observed level of significance is

$$2P(t_{10}\text{ random variable } > 2.49) = 2(\text{something between } .01 \text{ and } .025)$$

which is between .02 and .05, according to Table B-4. This is strong evidence that there is a difference in mean skid lengths.

(c) Use equation (6-35). For 95% confidence, the appropriate $t$ is $t = Q_{10(.975)} = 2.228$ from Table B-4, and the resulting interval is
\[ 384.83 - 359.83 \pm 2.228(17.377) \sqrt{\frac{1}{6} + \frac{1}{6}} = 25.0 \pm 22.3529 \]
\[ = [2.65, 47.35]. \]

(d) \[
\hat{\nu} = \]

\[
\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2 \text{ divided by } \left( \frac{s_1^4}{(n_1 - 1)n_1^2} + \frac{s_2^4}{(n_2 - 1)n_2^2} \right)
\]

\[ = \left[ \frac{236.567}{6} + \frac{367.367}{6} \right]^2 \text{ divided by} \]

\[
(236.567)^2/(5)(36) + (367.367)^2/(5)(36) \text{ gives}
\]

\[ \hat{\nu} = \frac{10,131.56}{1,060.68} = 9.55 \]

\[
\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^{1/2} = \left[ \frac{236.567}{6} + \frac{367.367}{6} \right]^{1/2} = 10.033
\]

Let the df be 10.

\[
(x_1 - x_2) \pm \hat{i} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \quad \hat{i} = 2.228(10 \text{df}). \quad \text{Thus,}
\]

\[
(384.833 - 359.833) \pm (2.228)(10.033)
\]

\[ 25 \pm 22.3535 \text{ gives } [2.65, 47.35]. \quad \text{Using 9 df., } \hat{i} = 2.262(9 \text{df}) \]

\[ \text{and the interval becomes } [2.3, 47.7]. \]
Solutions for Section 6.4 all

1. (a) Use equation (6-42) and Table B-5. For a 95% two-sided interval, \( U = Q_6(.975) = 12.833 \) and \( L = Q_6(.025) = .831 \). The resulting interval for \( \sigma^2 \) is \([92.17123, 1423.385]\); taking the square root of each endpoint, the interval for \( \sigma \) is \([9.60, 37.73]\) cm.

(b) For a 99% one-sided interval, \( L = Q_6(.01) = .554 \) and the interval for \( \sigma^2 \) is \([-\infty, 3315.584]\); taking the square root, the interval for \( \sigma \) is \([-\infty, 57.58]\) cm.

(c) 1. \( \text{Ho: } \frac{s_1^2}{s_2^2} = 1 \).
   2. \( \text{Ha: } \frac{s_1^2}{s_2^2} \neq 1 \).
   3. The test statistic is given by equation (6-49) with \( \# = 1 \), and the reference distribution is the \( F_{5,5} \) distribution. Small or large observed values of \( F \) (relative to 1) will be considered as evidence against \( \text{Ho} \).
   4. The sample gives

\[
f = .644.
\]

5. The observed level of significance is

\[2P(\text{an } F_{5,5} \text{ random variable < .644}).\]

It is necessary to switch the degrees of freedom, invert the observed \( f \), and change the inequality to find the probability to the left of this small quantile using Tables B-6.

(Switching the degrees of freedom has no effect here, since the degrees of freedom are the same.)

\[
= 2P(\text{an } F_{5,5} \text{ random variable } > \frac{1}{.644})
= 2P(\text{an } F_{5,5} \text{ random variable } > 1.55)
= 2(\text{something greater than } .25),
\]

so the p-value is greater than .5, according to Tables B-6. There is no evidence of a difference in variability between treaded and smooth stopping distances.

(d) Use equation (6-47) and Tables B-6. For 90% confidence, \( U = Q_{6.5}(.95) = 5.05 \) and \( L = Q_{6.5}(.05) = \frac{1}{Q_{6.5}(95)} = \frac{1}{5.05} \). The resulting interval for \( s_x^2 \) is \([.1275153, 3.25196]\). Taking the square root of each endpoint, the interval for \( s_x \) is \([.357, 1.803]\).

2. (a) \([\sqrt{(n-1)s^2/U}, +\infty]\) which becomes \([\sqrt{14/23.685(9.67323)}, +\infty]\) or \([7.437, +\infty]\) is a lower one-sided 95% confidence interval for the standard deviation of the top bolt torques.

(b) \([6\sqrt{(n-1)s^2/U}, +\infty]\) becomes \([44.622, +\infty]\), a lower one-sided 95% confidence interval for 6\( \sigma \).

(c) Torque of the top bolt is not independent of the torque on the bottom bolt for a given piece.
Solutions for Section 6.5 all

1. (a) Using equation (6-57), the appropriate \( z \) for 95% confidence is 1.96. The resulting interval is

\[
.66 \pm 1.96 \frac{1}{\sqrt{100}} = .66 \pm .096
\]

\[
= [.564, .758].
\]

For a 95% one-sided interval, construct a 90% two-sided interval and use the lower endpoint. The appropriate \( z \) for a 90% two-sided interval is 1.645, so the 95% one-sided interval is

\[
.66 - 1.645 \frac{1}{2\sqrt{100}} = .66 - .08225
\]

\[
= .578.
\]

Using equation (6-59), the appropriate \( z \) for 95% confidence is 1.96. The resulting interval is

\[
.66 \pm 1.96 \sqrt{.66(1-.66)} = .66 \pm .0928
\]

\[
= [.567, .753].
\]

A 95% one-sided interval is

\[
.66 - 1.645 \sqrt{.66(1-.66)} = .66 - .077925
\]

\[
= .582.
\]

The two different methods give similar results, because \( \hat{p} = .66 \) is close to \( \frac{1}{2} \).

(b) 1. \( H_0: p = .55 \).

2. \( H_a: p > .55 \).

3. The test statistic is given by equation (6-53) with \( \# = .55 \), and the reference distribution is the standard normal distribution. Observed values of \( Z \) far above zero will be considered as evidence against \( H_0 \).

4. The sample gives

\[
z = 2.21.
\]

5. The observed level of significance is

\[
P(\text{a standard normal random variable} > 2.21) = P(\text{a standard normal random variable} < -2.21) = .0136
\]

using Table B-3. This is strong evidence of an improvement in yield.
(c) Label the small shot size Sample 1 and the large shot size Sample 2. Using equation (6-65), the appropriate \( z \) for 95% confidence is 1.96. The resulting interval is

\[
.66 - .53 \pm 1.96 \left( \frac{1}{2} \right) \sqrt{\frac{1}{100} + \frac{1}{100}} = .13 \pm .13859
\]

\[
= [-.0086, .2686].
\]

Using equation (6-67), the resulting interval is

\[
.66 - .53 \pm 1.96 \sqrt{\frac{.66(1-.66)}{100} + \frac{.53(1-.53)}{100}} = .13 \pm .13487
\]

\[
= [-.00487, .2649].
\]

Both methods show that there is some evidence that the fraction conforming is larger for the small shot size, but the evidence is not conclusive.

(d) 1. \( H_0: p_1 - p_2 = 0 \).
2. \( H_a: p_1 - p_2 \neq 0 \).
3. The test statistic is given by equation (6-72), and the reference distribution is the standard normal distribution. Observed values of \( Z \) far above or below zero will be considered as evidence against \( H_0 \).

4. The sample gives

\[
z = 1.87
\]

5. The observed level of significance is

\[
2P(\text{a standard normal random variable} > 1.87)
\]

\[
= 2P(\text{a standard normal random variable} < -1.87)
\]

\[
= 2(.0307) = .0614.
\]

using Table 6-3. This is moderate evidence that the shot size affects the fraction of pellets conforming.

2. To ensure that the sample size is large enough (no matter what \( p \) really is), assume that \( p = .5 \) and use the conservative interval given by equation (6-57). For 95% confidence, \( z = 1.96 \), so

\[
\Delta = 1.96 \frac{1}{\sqrt{n}}.
\]

We want this to be less than or equal to .01. Solving the inequality for \( n \) gives \( n \geq 9804 \). Pollsters use \( \Delta = .03 \), resulting in \( n = 1068 \), which is the minimum sample size that you will usually see when the “margin of error” is \( \pm3\% \).

3. Using equation (6-57), the appropriate \( z \) for 99% confidence is 2.58. The resulting interval is

\[
\frac{405 - 290}{405} \pm 2.58 \frac{1}{2\sqrt{405}} = .28395 \pm .064101
\]

\[
= [.220, .348].
\]
Using equation (6-59), the appropriate $z$ for 99% confidence is 2.58. The resulting interval is

$$0.28395 \pm 2.58 \sqrt{\frac{0.28395(1 - 0.28395)}{405}} = 0.28395 \pm 0.057808$$

$$= [0.226, 0.342].$$

4. 1. $H_0$: $p_1 - p_2 = 0$.
2. $H_a$: $p_1 - p_2 \neq 0$.

3. The test statistic is given by equation (6-70), and the reference distribution is the standard normal distribution. Observed values of $Z$ far above or below zero will be considered as evidence against $H_0$.

4. The sample gives

$$z = -0.97$$

5. The observed level of significance is

$$2P(a \text{ standard normal random variable } < -0.97) = 2(0.1660) = 0.3320.$$ 

using Table B-3. There is little or no evidence of a difference in machine nonconforming rates. This suggests that large sample sizes are needed to detect even moderate differences in underlying proportions. In general, large samples are needed to make definitive conclusions based on qualitative data.
Solutions for Section 6.6, Exercises 2 and 3.

2. (a) $\bar{x} \pm t_s \sqrt{1 + 1/n}$ becomes $215.1 \pm (1.833)(42.943)(1 + 1/10)^{1/2}$ or
(132.543, 297.656) is a two-sided 90% prediction interval for an additional spring lifetime under this stress.

(b) $\bar{x} \pm t_{2s}$ or $215.1 \pm (2.856)(42.943)$. Thus, (92.455, 337.745) includes 90% of the population with 95% confidence.

(c) The 95% tolerance interval for 90% of the population (interval in (b)) is much wider than the 90% prediction interval for the next observation (interval in (a)). The interval in (b) contains 90% of future observations with 95% confidence. In repeated applications, the interval constructed as in (a) will contain an average of 90% of future observations. Any one interval constructed like the one in (a) may contain less or more than 90% of all future observations.

(d) The 90% interval for the mean lifetime (900 N/m²) is shorter than both intervals given respectively in (a) and (b). The 90% interval for the mean lifetime (at 900 N/m²) is such that 90% of all intervals similarly constructed from samples of size $n = 10$ will cover or include the true average lifetime (at 900 N/m² stress).

(e) $\bar{x} - t_s \sqrt{1/n}$ produces $215.1 - (1.383)(42.943) \sqrt{1.1} = 152.811$

Thus, $[152.811, + \infty]$ is the 90% confidence lower one-sided prediction interval for an additional spring lifetime under this stress.

(f) $\bar{x} - t_s \sqrt{n} = 215.1 - (2.355)(42.943) = 215.1 - 101.13 = 113.969$. Thus, $[113.969, + \infty]$ is a 95% lower tolerance bound for 90% of all spring lifetimes under this stress.

3. (a) Use equation (6-83). With 99% confidence, $n \approx 25$, and $p = .90$, the appropriate value for $t_2$ is 2.506 (see Table B-7-A). The interval is then

$$4.9 \pm 2.506(.59) = 4.9 \pm 1.47854 = [3.42146, 6.37854].$$

Exponentiating each endpoint, $[30.61, 589.07]$ is a 99% tolerance interval for 90% of additional raw aluminum contents.
(b) Use equation (6-73). For 90% confidence, the appropriate $t$ is $t = Q_{25}(.95) = 1.708$, from Table B-4. The resulting interval is

$$4.9 \pm 1.708(.59)\sqrt{1 + \frac{1}{26}} = 4.9 \pm 1.026916$$

$$= [3.873084, 5.926916].$$

Exponentiating each endpoint, $[48.09, 375.00]$ is a 90% prediction interval for a single additional raw aluminum content.

(c) The interval in (a) is wider than the interval in (b). This is usually true when applying tolerance intervals (with large $p$) and prediction intervals in the same situation, with similar confidences.