There are 10 questions on the following 5 pages. Do as many of them as you can in the available time. I will score each question out of 10 points AND TOTAL YOUR BEST 7 SCORES. (That is, this is a 70 point exam.)
1. Random variables $x$ and $y$ have a joint probability mass function specified in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.08</td>
<td>.10</td>
<td>.12</td>
<td>.10</td>
</tr>
<tr>
<td>4</td>
<td>.08</td>
<td>.10</td>
<td>.12</td>
<td>.10</td>
</tr>
<tr>
<td>6</td>
<td>.08</td>
<td>.10</td>
<td>.12</td>
<td>.10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.06</td>
<td>.04</td>
<td>.11</td>
<td>.09</td>
</tr>
<tr>
<td>3</td>
<td>.06</td>
<td>.04</td>
<td>.11</td>
<td>.09</td>
</tr>
<tr>
<td>5</td>
<td>.06</td>
<td>.04</td>
<td>.11</td>
<td>.09</td>
</tr>
</tbody>
</table>

- **10 pts**
  a) Find the SEL optimal predictor of $y$ based on $x$, say $\hat{y}_{\text{SEL}}(x)$. (Specify the 4 values $\hat{y}_{\text{SEL}}^\text{opt}(1), \hat{y}_{\text{SEL}}^\text{opt}(2), \hat{y}_{\text{SEL}}^\text{opt}(3),$ and $\hat{y}_{\text{SEL}}^\text{opt}(4)$. There is no need to do the arithmetic.)

  We want the conditional mean function. That is

  \[
  \begin{align*}
  \hat{y}(1) &= 0\left(\frac{.06}{2}\right) + 1\left(\frac{.06}{2}\right) + 2\left(\frac{.06}{2}\right) \\
  \hat{y}(2) &= 0\left(\frac{.04}{2}\right) + 1\left(\frac{.04}{2}\right) + 2\left(\frac{.04}{2}\right) \\
  \hat{y}(3) &= 0\left(\frac{.11}{3}\right) + 1\left(\frac{.11}{3}\right) + 2\left(\frac{.11}{3}\right) \\
  \hat{y}(4) &= 0\left(\frac{.10}{3}\right) + 1\left(\frac{.10}{3}\right) + 2\left(\frac{.10}{3}\right)
  \end{align*}
  \]

- **10 pts**
  b) Find the 0-1 loss predictor of $y$ based on $x$, say $\hat{y}_{\text{0-1}}(x)$. (Specify the 4 values $\hat{y}_{\text{0-1}}^\text{opt}(1), \hat{y}_{\text{0-1}}^\text{opt}(2), \hat{y}_{\text{0-1}}^\text{opt}(3),$ and $\hat{y}_{\text{0-1}}^\text{opt}(4)$.) Here we want the conditional mode function. That is

  \[
  \begin{align*}
  \hat{y}(1) &= 2 \\
  \hat{y}(2) &= 2 \\
  \hat{y}(3) &= 2 \\
  \hat{y}(4) &= 0
  \end{align*}
  \]
c) Suppose that one begins a Gibbs SSS algorithm at \((x^0, y^0) = (1, 1)\). Give the distribution one uses to generate \(x^1\). Then, supposing that \(x^1 = 2\), give the distribution that one uses in order to generate \(y^1\).

For generating \(x^1\) we use the conditional distribution of \(x\) given \(y = 1\) i.e.

\[
\begin{array}{c|cccc}
  x & 1 & 2 & 3 & 4 \\
  f(x|y = 1) & .3 & .3 & .3 & .3 \\
\end{array}
\]

For generating \(y^1\) we use the conditional dsn of \(y\) given \(x = 2\) i.e.

\[
\begin{array}{c|cccc}
  y & 0 & 1 & 2 \\
  f(y|x = 2) & .5 & .2 & .3 \\
\end{array}
\]

\[10 \text{ pts}\]

d) Find the distribution of the sum \(t = x + y\). (List possible values and corresponding probabilities in a table below.)

Looking at the table for sums and probabilities we see that possible values are 1, 2, 3, 4, 5, 6 and adding probabilities we get:

\[
\begin{array}{c|cccccc}
  t & 1 & 2 & 3 & 4 & 5 & 6 \\
  p(t) & .06 & .12 & .15 & .32 & .21 & .10 \\
\end{array}
\]

\[10 \text{ pts}\]
2. In the planning of a large engineering project, the number of days required to complete one step in the project is modeled as a continuous random variable with cumulative distribution function

\[
F(x) = \begin{cases} 
0 & \text{if } x < 100 \\
\frac{(x-100)^3}{100^3} & \text{if } 100 \leq x \leq 200 \\
1 & \text{if } x > 200 
\end{cases}
\]

a) A simulation is going to be used to study the feasibility of the project and variables with this distribution function are needed. For \( U \sim U(0,1) \) give a function \( h(u) \) for which \( h(U) \) has the target distribution (has cdf \( F(x) \)).

For \( u \in (0,1) \)

\[
F^{-1}(u) = \text{solution to } u = \frac{(x-100)^3}{100^3}
\]

This is \( 100^3u = (x-100)^3 \)

\[
\sqrt[3]{100^3u} = x-100
\]

\[
x = 100 + \sqrt[3]{100^3u}
\]

\( \text{So } \) \( F^{-1}(u) = 100 + 100\sqrt[3]{u} \)

b) Approximate the probability that the sample mean of \( n = 16 \) simulated values from the distribution specified by \( F \) exceeds 180 days. (You may use without proof the fact that the distribution has mean 175 and standard deviation 19.4.)

\[
M_\bar{x} = 175 \quad \text{and} \quad \sigma_\bar{x} = \frac{\sigma}{\sqrt{n}} = \frac{19.4}{4.85} = 4.025 \quad \text{and the CLT says that the standardized version of } \bar{x} \text{ is approximately standard normal. So}
\]

\[
P \left[ \bar{x} > 180 \right] = P \left[ \frac{\bar{x} - 175}{4.025} > \frac{180 - 175}{4.025} \right]
\]

\( \approx 1 - \Phi \left( \frac{180 - 175}{4.025} \right) \)
3. The exponential distribution with mean $\mu$ has standard deviation $\mu$. If $x_1, x_2, \ldots$ are iid according to this distribution, what is an approximate distribution for
\[ \ln(\bar{x}_n) \]
(the natural logarithm of the sample mean of the first $n$ of these variables)?

\[ \bar{x}_n \text{ is approximately } N(\mu, \frac{\mu^2}{n}). \]
Then $\ln(\bar{x}_n)$ is approximately normal with mean $\ln(\mu)$ and standard deviation
\[ \left| \frac{d}{dy} \ln(y) \right| \mu \left( \frac{\mu}{\sqrt{n}} \right) \]
\[ = \left| \frac{1}{\mu} \right| \left( \frac{\mu}{\sqrt{n}} \right) \]
\[ = \frac{1}{\sqrt{n}} \]
(Note: BTW, that this is free of $\mu$)

4. If $x_1, x_2, \ldots, x_n$ are iid $N(\mu, \sigma^2)$ then we know that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$. Let # stand for the lower 2.5% point of the $\chi^2_{n-1}$ distribution and ## stand for the upper 2.5% point of the $\chi^2_{n-1}$ distribution. Use these elements and produce 95% confidence limits for $\sigma$. That is, give formulas for random variables $L$ and $U$ so that in this context
\[ P[L < \sigma < U] = .95 \]
for all possible $(\mu, \sigma^2)$ pairs. (Explain/make clear why your limits will work.)

\[ P \left[ \# \leq \frac{(n-1)s^2}{\sigma^2} \leq ## \right] \]
\[ \frac{(n-1)s^2}{##} < \sigma^2 < \frac{(n-1)s^2}{#} \]
\[ \sqrt{\frac{(n-1)s^2}{##}} < \sigma < \sqrt{\frac{(n-1)s^2}{#}} \]
So 95% confidence limits for $\sigma$ are
\[ L = s\sqrt{\frac{n-1}{##}} \text{ and } U = s\sqrt{\frac{n-1}{#}} \]
5. The Poisson($\lambda$) pmf is $f(x | \lambda) = \lambda^x \exp(-\lambda) / x!$ for non-negative integer $x$. The $\Gamma(\alpha, \beta)$ pdf is $g(\theta) = \theta^{\alpha-1} \exp(-\theta / \beta) / \beta^\alpha \Gamma(\alpha)$ for $\theta > 0$. (You may use this information in the following.)

In a Bayes model, $x_1, x_2, \ldots, x_n$ are iid Poisson($\lambda$) and a priori $\lambda \sim \text{Exp}(1)$. Identify the posterior distribution.

The likelihood times prior is

$$\prod_{i=1}^{n} \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!} \cdot \lambda^{x_i} \exp(-\lambda)$$

as a function of $\lambda$. This is proportional to

$$\lambda^{\sum x_i} \exp(- (n+1) \lambda)$$

This is proportional to the $\Gamma$ density for $\lambda$ with parameters $\alpha = \sum x_i + 1$ and $\beta = \frac{1}{n+1}$.

6. A single random observation $x$ has a distribution with pdf $f(x | \theta) = (x^\theta \cdot (\theta + 1)) [0 < x < 1]$.

Find the maximum likelihood estimator of $\theta$ in this statistical model and write out (but do not attempt to evaluate/simplify) a completely specified definite integral that gives the squared error loss risk function of this estimator, $R(\theta)$.

$f(x | \theta) = (\theta + 1) x^\theta$ and $\ln f(x | \theta) = \theta \ln x + \ln(\theta + 1)$

So $\frac{d}{d\theta} \ln f(x | \theta) = \ln x + \frac{1}{\theta + 1}$ and at a maximum of $f(x | \theta)$ or $\ln f(x | \theta)$ over $\theta$, $\frac{d}{d\theta} \ln f(x | \theta) = 0$.

i.e. $\theta + 1 = \frac{1}{\ln x}$ i.e. $\theta = -\frac{1}{\ln x} - 1$. So $\hat{\theta}_{\text{MLE}} = -\frac{1}{\ln x} - 1$.

This estimator has risk function

$$R(\theta) = \int_{0}^{1} (-\frac{1}{\ln x} - 1 - \theta)^2 (\theta + 1) x^\theta \, dx$$