Suppose \( \mathbf{\xi} = (y_1, y_2, \ldots, y_n) \) are the \( y_i \) iid \( \mathcal{N}(0, \nu) \) and consider testing \( H_0 : \nu = 1 \) vs \( H_a : \nu \neq 1 \).

Here

\[
\lambda_n(\mathbf{\xi}) = \ln f(\mathbf{\xi} \mid \hat{\nu}^\text{MLE}) - \ln f(\mathbf{\xi} \mid 1)
\]

\[
= -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \hat{\nu}^\text{MLE} - \frac{\sum y_i^2}{2 \hat{\nu}^\text{MLE}}
\]

\[
- \frac{n}{2} \ln (2\pi) - \frac{1}{2} \sum y_i^2
\]

\[
= -\frac{n}{2} \ln \hat{\nu}^\text{MLE} + \frac{n}{2} \left( \hat{\nu}^\text{MLE} - 1 \right)
\]

Consider the function \( g(\nu) = -\frac{n}{2} \ln \nu + \frac{n}{2} (\nu - 1) \)

\( g'(\nu) = \frac{n}{2} \left( -1 + \frac{1}{\nu} \right) \) and \( : g'(\nu) < 0 \) if \( \nu < 1 \)

and \( g'(\nu) > 0 \) if \( \nu > 1 \). Then since \( g(1) = 0 \) there are unique \( \nu^L < 1 \) and \( \nu^U > 1 \). That solve

\[
2 g(\nu) = \text{upper 5\% point of } \chi^2_1 \text{ df} = 3.841
\]

One can solve for these numerically and end with a likelihood ratio test of approximate size .05 of the form

\[
a(\mathbf{\xi}) = \frac{1}{2} \left[ \hat{\nu}^\text{MLE} < \nu^L \text{ or } \hat{\nu}^\text{MLE} > \nu^U \right]
\]

(If you are having trouble following this, make a few simulated data values and actually find \( \nu^L \) and \( \nu^U \).)