Stat 231 Problem Set 6 Solutions

Note: for most of the book problems, I borrowed the solution from instructor’s manual. You can tell the difference by the format of the text. Let me know if you would like me to expand on anything.

Exercise 6.1 (Devore 5.9).

a. \[ P(|X - Y| \leq 2) = \int \int \text{region III} f(x,y) \, dy \, dx = \int_{20}^{30} \int_{x-2}^{x+2} K(x^2 + y^2) \, dy \, dx \]
\[ = K \int_{20}^{30} x^2 \, dx + K \int_{20}^{30} y^2 \, dy = 10K \int_{20}^{30} x^2 \, dx + 10K \int_{20}^{30} y^2 \, dy \]
\[ = 20K \left( \frac{19000}{3} \right) \Rightarrow K = \frac{3}{380000} \]

b. \[ P(X < 26 \text{ and } Y < 26) = \int \int 26 K(x^2 + y^2) \, dy \, dx = 12K \int_{20}^{26} x^2 \, dx \]
\[ 4K \int_{20}^{26} = 38304K = .3024 \]

c.

\[ P(|X - Y| \leq 2) = \int \int \text{region III} f(x,y) \, dy \, dx = 1 - \int \int f(x,y) \, dy \, dx - \int \int f(x,y) \, dy \, dx \]
\[ = 1 - \int_{20}^{28} \int_{x+2}^{x+2} (Kx^2 + Ky^2) \, dy \, dx - \int_{22}^{30} \int_{20}^{30} (Kx^2 + Ky^2) \, dy \, dx \]
\[ = 1 - \int_{20}^{28} \left[ Kx^2y + \frac{Ky^3}{3} \right]_{x+2}^{x+2} \, dx - \int_{22}^{30} \left[ Kx^2y + \frac{Ky^3}{3} \right]_{20}^{30} \, dx \]
\[ = 1 - \int_{20}^{28} \left[ 30Kx^2 + \frac{K(30)^3}{3} \right] \, dx \]
\[ - \int_{22}^{30} \left[ 26992K \right] \, dx = 1 - 40576K - 40576K = 1 - 2 \cdot 40576 \cdot \frac{3}{380000} = .3593 \]
Exercise 6.2 (Devoer 5.10).

b. \[ P(5.25 \leq X \leq 5.75, 5.25 \leq Y \leq 5.75) = P(5.25 \leq X \leq 5.75) \cdot P(5.25 \leq Y \leq 5.75) = (\text{by independence}) (.5)(.5) = .25 \]

c. \[
\begin{align*}
\text{P}(X,Y \in A) &= \iiint_A dx dy \\
&= \text{area of } A = 1 - (\text{area of I + area of II}) \\
&= 1 - \frac{25}{36} = \frac{11}{36} = .306
\end{align*}
\]

Exercise 6.3 (Devoer 5.12).

a. \[ P(X > 3) = \int_3^\infty \int_0^\infty xe^{-x+y} dy dx = \int_3^\infty e^{-x} dx = .050 \]

b. The marginal pdf of X is \[ \int_0^\infty xe^{-x+y} dy = e^{-x} \text{ for } 0 \leq x \] that of Y is \[ \int_0^\infty xe^{-y} dx = \frac{1}{\left(1 + y\right)^2} \text{ for } 0 \leq y. \] It is now clear that \( f(x,y) \) is not the product of the marginal pdf’s, so the two r.v.’s are not independent.

c. \[
\begin{align*}
P(\text{at least one exceeds 3}) &= 1 - P(X \leq 3 \text{ and } Y \leq 3) \\
&= 1 - \int_0^3 \int_0^3 xe^{-x+y} dy dx - \int_0^3 xe^{-x} e^{-y} dy \\
&= 1 - \int_0^3 \left(1 - e^{-2x}\right) dx = e^{-3} + .25 - .25e^{-12} = .300
\end{align*}
\]
Exercise 6.4 (Devoer 5.13).

a. \( f(x,y) = f_x(x) \cdot f_y(y) = \begin{cases} e^{-x-y} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \)

b. \( P(X \leq 1 \text{ and } Y \leq 1) = P(X \leq 1) \cdot P(Y \leq 1) = (1 - e^{-1}) (1 - e^{-1}) = .400 \)

c. \( P(X + Y \leq 2) = \int_0^2 \int_0^{2-x} e^{-x-y} \, dy \, dx = \int_0^2 e^{-x} \left[ 1 - e^{-(2-x)} \right] \, dx \\
= \int_0^2 (e^{-x} - e^{-2}) \, dx = 1 - e^{-2} - 2e^{-2} = .594 \)

d. \( P(X + Y \leq 2) = \int_0^2 e^{-x} \left[ 1 - e^{-(1-x)} \right] \, dx = 1 - 2e^{-1} = .264, \)

so \( P(1 \leq X + Y \leq 2) = P(X + Y \leq 2) - P(X + Y \leq 1) = .594 - .264 = .330 \)

Exercise 6.5 (Devoer 5.15).

a. \( F(y) = P( Y \leq y ) = P \left[ (X_1 \leq y) \cup ((X_2 \leq y) \cap (X_1 \leq y)) \right] \\
= P( X_1 \leq y) + P( (X_2 \leq y) \cap (X_1 \leq y)) - P( (X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y)) \\
= (1 - e^{-\lambda y}) + (1 - e^{-\lambda y})^2 - (1 - e^{-\lambda y})^3 \text{ for } y \geq 0 \)

b. \( f(y) = F'(y) = \lambda e^{-\lambda y} + 2(1 - e^{-\lambda y}) \left( \lambda e^{-\lambda y} \right) - 3(1 - e^{-\lambda y})^2 \left( \lambda e^{-\lambda y} \right) \\
= 4\lambda e^{-2\lambda y} - 3\lambda e^{-3\lambda y} \text{ for } y \geq 0 \)

\[ E(Y) = \int_0^\infty y \cdot \left( 4\lambda e^{-2\lambda y} - 3\lambda e^{-3\lambda y} \right) \, dy = 2 \left( \frac{1}{2\lambda} \right) - \frac{1}{3\lambda} = \frac{2}{3\lambda} \]

Exercise 6.6 (Devoer 5.18).
Exercise 6.7 (Devoer 5.25).

\[ E(XY) = E(X) \cdot E(Y) = L \cdot L = L^2 \]

Exercise 6.8 (Devoer 5.26).

Revenue = 3X + 10Y, so E (revenue) = E (3X + 10Y)

\[ = \sum_{x=0}^{5} \sum_{y=0}^{2} (3x + 10y) \cdot p(x, y) = 0 \cdot p(0,0) + \ldots + 35 \cdot p(5,2) = 15.4 \]

Exercise 6.9 (Devoer 5.27).
Exercise 6.10 (Devoer 5.31).

a. \( E(X) = \int_0^3 xf(x)dx = \int_0^3 x[10Kx^2 + .05]dx = 25.329 = E(Y) \)
\( E(XY) = \int_0^3 \int_0^3 xy \cdot K(x^2 + y^2)dydx = 641.447 \)
\( \Rightarrow \text{Cov}(X,Y) = 641.447 - (25.329)^2 = -.111 \)

b. \( E(X^2) = \int_0^3 x^2 [10Kx^2 + .05]dx = 649.8246 = E(Y^2) \),
so \( \text{Var}(X) = \text{Var}(Y) = 649.8246 - (25.329)^2 = 8.2664 \)
\( \Rightarrow \rho = \frac{-\.111}{\sqrt{8.2664}(8.2664)} = -.0134 \)

Exercise 6.11 (Devoer 5.32).

\[ E(XY) = \int_0^\infty \int_0^\infty xy \cdot xe^{-x(1+y)}dydx = \cdots = 1. \]
Yet, since the marginal pdf of \( Y \) is \( \frac{1}{(1-y)^2} \)
for \( y = 0 \), \( E(Y) = \int_0^\infty \frac{y}{(1+y)^2}dy = \infty \). Therefore, \( \text{Cov}(X,Y) \) and \( \text{Corr}(X,Y) \) do not exist,
since they require this integral (among others) to be convergent.

Exercise 6.12 (Devoer 5.77).
a. \[ 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{20} \int_{20-x}^{20} kxy \, dy \, dx + \int_{20}^{30} \int_{0}^{20-x} kxy \, dy \, dx \]
\[ = \frac{81,250}{3} \cdot k \Rightarrow k = \frac{3}{81,250} \]

b. \[ f_X(x) = \begin{cases} \int_{20-x}^{20} kxy \, dy = k(250x - 10x^2) & 0 \leq x \leq 20 \\ \int_{0}^{20-x} kxy \, dy = k(450x - 30x^2 + \frac{1}{2}x^3) & 20 \leq x \leq 30 \end{cases} \]

and by symmetry \( f_Y(y) \) is obtained by substituting \( y \) for \( x \) in \( f_X(x) \). Since \( f_X(25) > 0 \), and \( f_Y(25) > 0 \), but \( f(25, 25) = 0 \), \( f_X(x) \cdot f_Y(y) = f(x, y) \) for all \( x, y \), so \( X \) and \( Y \) are not independent.
Exercise 6.13 (Devoe 5.92).

\[ E(Y) = 120 \left( \frac{1}{10} + \frac{1}{15} + \frac{1}{20} \right) = 26 \]

The partial derivatives of \( h(\mu_1, \mu_2, \mu_3, \mu_4) \) with respect to \( x_1, x_2, x_3, \) and \( x_4 \) are \( -\frac{x_4}{x_2^2}, \) \( -\frac{x_4}{x_2^2}, \) and \( \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, \) respectively. Substituting \( x_1 = 10, x_2 = 15, x_3 = 20, \) and \( x_4 = 120 \) gives -1.2, -0.5333, -3.000, and 0.2167, respectively, so \( \text{Var}(Y) = \left(1\right)(-1.2)^2 + \left(1\right)(-0.5333)^2 + \left(1\right)(-3.000)^2 + \left(1\right)(0.2167)^2 = 2.6783, \) and the approximate \( \text{sd} \) of \( y \) is 1.64.

Exercise 6.14 (Prof. Vardeman’s Ex. 2).

I want to simulate \( X, Y, \) and \( |X - Y| \) from Devore problem 5.10. I add 100 rows like so:
I give the column the name, "X", and I simulate it by right clicking the title area selecting "Formula":
I locate the red box (which initially has "no formula" written inside), and I write Random Uniform() + 5:

I do the same for Y (make a new column called "Y"). Now, I put \(|X - Y|\) in a third column:
Now, I answer part A by going to Analyze → Distribution in the top menu:

I drag "—X-Y—" from the "Select Columns" area on the left into the white box directly to the right of the "Y, Columns" button. Then, I click "ok":

10
You can now read that the mean of the "—X-Y—" data is \[0.3341108\]. This is my estimate of \(E(|X - Y|)\). Yours should be close.

For part B, I click the red triangle next to "—X-Y—" at the top of the output, and I click on "CDF Plot".
You need to read off the cdf value at $|X - Y| = 1/6 \approx .1667$. To do this, use the crosshair tool. You can select it at the top of the output window. I have it selected in the following screenshot:

Click on the plot area with the crosshair tool as close as you can to $|X - Y| = .1667$, and voila:

I got $0.2889$ as my answer.
**Exercise 6.15** (Prof. Vardeman’s Ex. 3).
I start with a column named "X+Y" with, say, 1000 rows. (I walk you through how to add rows at the beginning of problem 2 above):

I make a formula for "X+Y" just like in problem 2: I right click at the top of the column and select "Formula":

Now, to simulate the standard exponential, I select the drop-down menu at the upper right of the function window and select "Random":

I select Random Exp, drag it to the red box that initially says "no formula".

I just learned that Random Exp() is the function I need to sample from a standard exponential distribution. However, I want "X+Y" to be the sum of two independent random samples of standard exponential distributions. Hence, in that little red box, I need to write "(Random Exp())+(Random Exp()):

I click "OK", and voila:
As I did in problem 2, I go to Analyze → Distribution and add a CDF plot:

Let $F$ be the cdf of $X + Y$. Using the crosshairs method that I showed for problem 2, I get $F(2) = .5833$ and $F(1) = .2556$. Problem 5.13c asks for $F(2) = .5833$, and problem 5.13d asks for $F(2) - F(1) = .5833 - .2556 = .3277$. 

15
Since the pdf is not strictly decreasing, \(X + Y\) cannot have an exponential distribution. In fact, \(X + Y\) has a gamma distribution with parameters \(\alpha = 2\) and \(\beta = 1\). To show this, I calculate the following for \(z > 0\) (use the independence of \(X\) and \(Y\)).

\[
f_{X+Y}(z) = P(X + Y = z) = \int_0^z f_{X,Y}(X = w, Y = z - w)dw = \int_0^z f_X(w)f_Y(z - w)dw
\]

\[
= \int_0^z e^{-w}e^{-(z-w)}dw = \int_0^z e^{-z}dw = [we^{-z}]_{w=0}^{w=z} = ze^{-z}
\]

Hence, the pdf of \(X + Y\) at \(z\) is 0 when \(z < 0\) and \(ze^{-z}\) when \(z \geq 0\). This is exactly the pdf of Gamma\((\alpha = 2, \beta = 1)\). Look in your book (section 4.4) if you don’t believe me.

**Exercise 6.16** (Prof. Vardeman’s Ex. 4).
Set up a data table with one column, ”Y”, and 1000 rows. Pull up the formula menu for column ”Y” by right-clicking the top of the ”Y” column like I did in problems 2 and 3 above. You should end up here:
The formula you want to write should be equivalent to $\min(X_1, \max(X_2, X_3))$, where the $X_i$'s are independent exponential random variables with parameter $\lambda = 1$. There are functions for minimum, maximum, and random exponential in the drop-down menu that I expanded here:

Do some noodling around under the "Random" and "Statistical" groups. You will find that the formula you need is "$\min((\text{Random Exp}()), \max((\text{Random Exp}()), (\text{Random Exp}())))$":

As in problems 2 and 3, I use Analyze ¿ Distribution to get the pdf and the mean:
From the output, the mean of the simulated values of $Y$ is $0.6571684$, which is my estimate for $E(Y)$. Yours should be similar. Also, the empirical pdf in the screenshot above looks roughly exponential.