Problem Set 3 Solutions

Exercise 3.1 (Devore 2.29).

a. Domain names with just 2 letters: this is like sampling from a bag of 26 unique blocks with replacement. \# names = (# letters) \times (# letters) = 26 \times 26 = \boxed{676}.

Domain names of length 2 with digits and letters: this time we have 36 things to choose from (26 letters, 10 digits. \# names = 36 \times 36 = \boxed{1296})

b. This is exactly like part (a) except that now, domain names are 3 characters long instead of 2. Without digits: \# names = 26^3 = \boxed{17,576}

With digits: \# names = 36^3 = \boxed{46,656}.

c. 26^4 = \boxed{456,976} \cdot 36^4 = \boxed{1,679,616}

d. probability that a 4-character name is already taken = 1 - probability that a name is available = 1 - \frac{\# available names}{\# allowable names} = 1 - \frac{97786}{36^4} = \boxed{.942}

Exercise 3.2 (Devore 2.32).

a. 5 for receiver \times 4 for disc player \times 3 for speakers \times 4 for turntable = \boxed{240}

b. 1 \times 1 \times 3 \times 4 = \boxed{12}

c. 4 \times 3 \times 3 = \boxed{108}

d. \# ways with at least one Sony component = (total \# ways) - (\# with no Sony component) = 240 - 4 \times 3 \times 3 = \boxed{132}

e. P(at least one Sony component) = \frac{\# ways to select at least one Sony component}{total \# possible selections} = \frac{132}{240} = \boxed{.55}

P(exactly one sony component) = P(the only Sony component is the receiver) + P(the only Sony component is the compact disc player) + P(the only Sony component is the turntable) = \frac{1 \times 4 \times 3 \times 3}{240} + \frac{4 \times 3 \times 3}{240} + \frac{4 \times 3 \times 1}{240} = \boxed{.413}
Exercise 3.3 (Devore 2.34).

(a) \( \binom{25}{5} = \frac{25!}{5!20!} = 53,130 \)

(b) \# ways = (# ways to select 2 with electrical defects) \times (# ways to select 3 without electrical defects) = \( \frac{6!}{2!4!} \cdot \frac{19!}{16!3!} = 14535 \)

(c) \( \text{P(at least 4)} = \text{P(exactly 4)} + \text{P(exactly 5)} = \frac{\binom{19}{4} \cdot \binom{6}{1}}{\binom{25}{5}} + \frac{\binom{19}{5} \cdot \binom{6}{0}}{\binom{25}{5}} = 0.6565 \)

Exercise 3.4 (Devore 2.38).

(a) \( \text{P(select 2 75-W bulbs)} = \frac{\binom{6}{2} \cdot \binom{9}{1} \cdot \binom{15}{3}}{\binom{15}{3}} = .2967 \)

(b) \( \text{P(same rating)} = \text{P(select all 40-W)} + \text{P(select all 60-W)} + \text{P(select all 75-W)} = \frac{\binom{4}{3} \cdot \binom{15}{3}}{\binom{15}{3}} + \frac{\binom{5}{3} \cdot \binom{15}{3}}{\binom{15}{3}} + \frac{\binom{6}{3} \cdot \binom{15}{3}}{\binom{15}{3}} = .0747 \)

(c) \( \frac{\binom{4}{1} \cdot \binom{5}{1} \cdot \binom{6}{1}}{\binom{15}{3}} = .2637 \)

(d) \( \text{P(examine at least six bulbs)} = \text{P(draw 5 non-75-W bulbs)} = \frac{\binom{4+5}{5}}{\binom{15}{5}} = .042 \)

Exercise 3.5 (Devore 2.71).

(a) Since \( A \) and \( B \) are independent \( A' \) and \( B' \) are independent (see page 83 just below Equation 2.7). Hence, \( \text{P}(B' \mid A') = P(B') = 1 - P(B) = 1 - .7 = .3 \)

(b) \( \text{P}(A \cup B) = P(A) + P(B) - P(A \cap B) \overset{\text{independence}}{=} P(A) + P(B) - P(A)P(B) = .4 + .7 - .4 \cdot .7 = .82 \)

(c) \( \text{P}(A \cap B' \mid A \cup B) = \frac{\text{P}(A \cap B')}{\text{P}(A \cup B)} = \frac{P(A \cap B')}{P(A\cup B)} \overset{\text{independence}}{=} \frac{\text{P}(A)P(B')}{P(A \cup B)} = .4 \cdot .7 \cdot .82 = .146 \)

Exercise 3.6 (Devore 2.77).
a. Let $p$ be the probability that a given rivet is defective. Then:

\[.2 = P(\text{seam is defective}) = 1 - P(\text{seam is good}) = 1 - P(\text{all 25 rivets are good}) = 1 - P(\text{rivet 1 is good}) \cap \cdots \cap P(\text{rivet 25 is good}) = 1 - (1 - p)^{25}\]

Hence, $p = 0.00889$

b. Use the calculation above: if only 10% of the seams need reworking, then

\[.1 = 1 - (1 - p)^{25}, \text{ so } p = 0.00421\]

Exercise 3.7 (Devore 2.78).

\[P(\text{at least one valve opens}) = 1 - P(\text{no valve opens}) = 1 - (1 - 0.95)^5 \approx 1 - 0.9999969 = 0.99999969\]

Exercise 3.8 (Devore 2.79).

Let $O$ be the event that the older pump fails and $N$ be the event that the newer pump fails. We are given that $P(O \cap N') = .1$ and $P(O' \cap N) = .05$ (look carefully at the prompt. For each pump, it gives the probability that ONLY that pump will fail). By the Law of Total Probability,

\[P(O) = P(O \cap N) + P(O \cap N') = x + .1\]
\[P(N) = P(O \cap N) + P(O' \cap N) = x + .05\]

where $x = P(O \cap N)$. Since the pumps fail independently of one another,

\[x = P(O \cap N) = P(O) \cdot P(N) = (x + .1)(x + .05)\]
\[\Rightarrow x^2 - .85x + .005 = 0\]

We solve the above quadratic equation to get $x = .0059$ or $x = .8441$. Hence, $P(O \cap N) = .0059$ or .8441. Hopefully, the true system failure probability is the smaller of the two, but we can’t know with the information given.

Exercise 3.9 (Devore 2.80).

Let $C_i$ be the event that the $i$'th component FAILS. Then, $P(\text{system works}) =$
1 - P(system fails) =

\[ 1 - P(C_1 \cap C_2 \cap (C_3 \cup C_4)) \]

\[ = 1 - P(C_1)P(C_2)P(C_3 \cup C_4) \]

\[ = 1 - P(C_1)P(C_2)[P(C_3) + P(C_4) - P(C_3 \cap C_4)] \]

\[ = 1 - P(C_1)P(C_2)[P(C_3) + P(C_4) - P(C_3)P(C_4)] \]

\[ = 1 - .1 \cdot .1 \cdot (.1 + .1 - .1 \cdot .1) \]

\[ = .9981 \]

**Exercise 3.10** (Devore 3.11).

a.

<table>
<thead>
<tr>
<th>x</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(x)</td>
<td>.45</td>
<td>.40</td>
<td>.15</td>
</tr>
</tbody>
</table>

b.

<table>
<thead>
<tr>
<th>x</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.5</td>
<td>0.4</td>
<td>.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

**Exercise 3.11** (Devore 3.12).

a. P(everyone is accommodated) =

\[ P(Y \leq 50) = .05 + .1 + .12 + .14 + .25 + .17 = .83 \]
b. \( P(\text{not everyone is accommodated}) = 1 - P(\text{everyone is accommodated}) = 1 - .83 = .17 \)

c. \( P(\text{first person on standby gets a seat}) = P(Y \leq 49) = .05 + .1 + .12 + .14 + .25 = .66 \) Assuming the first and second people on standby actually take seats if they are available, 
\( P(\text{third person on standby gets a seat}) = P(Y \leq 47) = .05 + .1 + .12 = .27 \),

Exercise 3.12 (Devore 3.13).

a. \( P(X \leq 3) = p(0) + p(1) + p(2) + p(3) = .10 + .15 + .20 + .25 = .70 \)

b. \( P(X < 3) = P(X \leq 2) = p(0) + p(1) + p(2) = .45 \)

c. \( P(X \geq 3) = p(3) + p(4) + p(5) + p(6) = .55 \)

d. \( P(2 \leq X \leq 5) = p(2) + p(3) + p(4) + p(5) = .71 \)

e. The number of lines not in use is \( 6 - X \), so we calculate \( P(2 \leq 6 - X \leq 4) = P(-4 \leq X - 6 \leq -2) = P(2 \leq X \leq 4) = p(2) + p(3) + p(4) = .65 \)

f. \( P(6 - X \geq 4) = P(6 \geq X + 4) = P(2 \geq X) = p(0) + p(1) + p(2) = .45 \)

Exercise 3.13 (Devore 3.22).

\[
\begin{align*}
F(0) &= P(X \leq 0) = .1 \\
F(1) &= P(X \leq 1) = .1 + .15 = .25 \\
F(2) &= P(X \leq 2) = .1 + .15 + .2 = .45 \\
F(3) &= P(X \leq 3) = .1 + .15 + .2 + .25 = .7 \\
F(4) &= P(X \leq 4) = .1 + .15 + .2 + .25 + .2 = .9 \\
F(5) &= P(X \leq 5) = .1 + .15 + .2 + .25 + .2 + .06 = .96 \\
F(6) &= P(X \leq 6) = .1 + .15 + .2 + .25 + .2 + .06 + .04 = 1
\end{align*}
\]

Hence:

<table>
<thead>
<tr>
<th>( F(x) )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.00</td>
<td>( x &lt; 0 )</td>
</tr>
<tr>
<td>.10</td>
<td>( 0 \leq x &lt; 1 )</td>
</tr>
<tr>
<td>.25</td>
<td>( 1 \leq x &lt; 2 )</td>
</tr>
<tr>
<td>.45</td>
<td>( 2 \leq x &lt; 3 )</td>
</tr>
<tr>
<td>.70</td>
<td>( 3 \leq x &lt; 4 )</td>
</tr>
<tr>
<td>.90</td>
<td>( 4 \leq x &lt; 5 )</td>
</tr>
<tr>
<td>.96</td>
<td>( 5 \leq x &lt; 6 )</td>
</tr>
<tr>
<td>1.00</td>
<td>( 6 \leq x )</td>
</tr>
</tbody>
</table>
And:

a. \( P(X \leq 3) = F(3) = 0.70 \)
b. \( P(X < 3) = P(X \leq 2) = F(2) = 0.45 \)
c. \( P(3 \leq X) = 1 - P(X \leq 2) = 1 - F(2) = 1 - 0.45 = 0.55 \)
d. \( P(2 \leq X \leq 5) = F(5) - F(1) = 0.96 - 0.25 = 0.71 \)

Exercise 3.14 (Devore 3.23).
Using the shortcut formula for variances, 
\[ (50 - 47)^2 \cdot p(47) + (51 - 47)^2 \cdot p(50) + (52 - 47)^2 \cdot p(51) + (53 - 47)^2 \cdot p(52) + (54 - 47)^2 \cdot p(53) + (55 - 47)^2 \cdot p(54) = \frac{1}{2} \cdot (25 + 2 + 1 + 2) = 39.557 \]

Exercise 3.15 (Devore 3.29).

a. \[ E(X) = \sum x \cdot p(x) = (1)(.05) + (2)(.1) + (4)(.35) + (8)(.4) + (16)(.1) = 6.45 \]

b. \[ V(X) = \sum (x - E(X))^2 \cdot p(x) = (1 - 6.45)^2 \cdot (.05) + (2 - 6.45)^2 \cdot (.1) + (4 - 6.45)^2 \cdot (.35) + (8 - 6.45)^2 \cdot (.4) + (16 - 6.45)^2 \cdot (.1) = 15.6475 \]

c. \[ \sigma_\chi = \sqrt{\chi} = 3.9557 \]

d. First, we compute \[ E(X^2) = \sum x^2 \cdot p(x) = (1^2)(.05) + (2^2)(.1) + (4^2)(.35) + (8^2)(.4) + (16^2)(.1) = 57.25 \] Next, we use the shortcut formula to calculate \[ V(X) = E(X^2) - [E(X)]^2 = 57.25 - 6.45^2 = 15.6475 \]

Exercise 3.16 (Devore 3.31).

\[ E(Y) = \sum y \cdot p(y) = (45)(.05) + (46)(.1) + (47)(.12) + (48)(.14) + (49)(.25) + (50)(.17) + (51)(.06) + (52)(.05) + (53)(.03) + (54)(.02) + (55)(.01) = 48.84 \]

\[ E(Y^2) = \sum y^2 \cdot p(y) = (45^2)(.05) + (46^2)(.1) + (47^2)(.12) + (48^2)(.14) + (49^2)(.25) + (50^2)(.17) + (51^2)(.06) + (52^2)(.05) + (53^2)(.03) + (54^2)(.02) + (55^2)(.01) = 2389.84 \]

Using the shortcut formula for variances, \[ V(Y) = E(Y^2) - [E(Y)]^2 = 2389.84 - 48.84^2 = 4.4944 \] Also, \[ \sigma_Y = \sqrt{V(Y)} = 2.12 \]

Now, if a value \( y \) is within one standard deviation of the mean of \( Y \), then \[ E(Y) - \sigma_Y \leq y \leq E(Y) + \sigma_Y = 48.84 - 2.12 \leq y \leq 48.84 + 2.12 = 46.72 \leq y \leq 50.96 \]

Since \( y \) only takes on the integers 45 through 55, this means \[ 47 \leq y \leq 50 \]. Hence, \( P(Y \) falls within one standard deviation of its mean) = \[ P(47 \leq Y \leq 50) = p(47) + p(48) + p(49) + p(50) = .12 + .14 + .25 + .17 = .68 \]

Exercise 3.17 (Devore 3.35).

If the store owner stocks 3 copies, then the profit \( h_3(x) \) (sales revenue - cost) is \[ 4x - 2 \cdot 3 = 4x - 6 \] for \( x = 1, 2, 3 \). Since the owner can only sell as much as he stocks, \( h_3(x) = 4 \cdot 3 - 2 \cdot 3 = 6 \) for \( x = 4, 5, 6 \). Similarly, if the owner stocks 4 copies, then the profit \( h_4(x) \) is \[ 4 \cdot x - 2 \cdot 4 = 4x - 8 \] for \( x = 1, 2, 3, 4 \) and \[ 4 \cdot 4 - 2 \cdot 4 = 8 \] for \( x = 5, 6 \). To summarize:
If \( Y = h_3(X) \), then 

\[
\]

If \( Z = h_4(X) \), then 

\[
\]

Since \( E(Y) < E(Z) \), the owner should probably stock 4 copies instead of 3.

**Exercise 3.18** (Devore 3.36).
Let \( H(X) \) be the company’s profit without the premium: \( D(X) - X \), where \( D \) is the deductible. Since the consumer pays no deductible if there is no accident and pays the full deductible if total damage is at least $500,

<table>
<thead>
<tr>
<th>( x )</th>
<th>( p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/15</td>
</tr>
<tr>
<td>2</td>
<td>2/15</td>
</tr>
<tr>
<td>3</td>
<td>3/15</td>
</tr>
<tr>
<td>4</td>
<td>4/15</td>
</tr>
<tr>
<td>5</td>
<td>3/15</td>
</tr>
<tr>
<td>6</td>
<td>2/15</td>
</tr>
</tbody>
</table>

\[
E[H(X)] = (0)(.8) + (-500)(.1) + (-4500)(.08) + (-9500)(.02) = -600.
\]

Hence, if the company wants a profit of $100, it should charge an annual premium of $700.

**Exercise 3.19** (Devore 3.39).
\[
E(X) = (1)(.2) + (2)(.4) + (3)(.3) + (4)(.1) = 2.3
\]

\[
E(X^2) = (1^2)(.2) + (2^2)(.4) + (3^2)(.3) + (4^2)(.1) = 6.1
\]

\[
V(X) = E(X^2) - E^2(X) = 6.1 - 2.3^2 = .81
\]

The number of pounds left is \( H(X) = 100 - 5X \), so

\[
E[H(X)] = E(100 - 5X) = 100 - 5E(X) = 100 - 5 \cdot 2.3 = 88.5 \quad \text{and}
\]

\[
V[H(X)] = V(100 - 5X) = 5^2V(X) = 5^2 \cdot .81 = 20.25
\]