ATTENTION!

Incorrect numerical answers unaccompanied by supporting reasoning will receive NO partial credit.

Correct numerical answers to difficult questions unaccompanied by supporting reasoning may not receive full credit.

SHOW YOUR WORK/EXPLAIN YOURSELF!

Completely absurd answers (that fail basic sanity checks but that you don't identify as clearly incorrect) may receive negative credit.
1. Random variables $X$ and $Y$ have a joint probability mass function specified in the table below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.08</td>
<td>.10</td>
<td>.12</td>
<td>.10</td>
</tr>
<tr>
<td>1</td>
<td>.06</td>
<td>.04</td>
<td>.11</td>
<td>.09</td>
</tr>
<tr>
<td>0</td>
<td>.06</td>
<td>.06</td>
<td>.07</td>
<td>.11</td>
</tr>
</tbody>
</table>

\[ x \]

\[ x-y=1 \]

\[ y=xy \]

| 2 | 2 | 3 | 3 |

7 pts a) Find the marginal pmf of the random variable $X$, $g(x)$. Specify this in tabular form (giving possible values of $X$ and corresponding probabilities $g(x)$).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>2</td>
<td>.2</td>
</tr>
<tr>
<td>3</td>
<td>.3</td>
</tr>
<tr>
<td>4</td>
<td>.3</td>
</tr>
</tbody>
</table>

7 pts b) Find $P[X-Y=1]$.

\[
P(X-Y=1) = f(3,2) + f(2,1) + f(1,0)
\]

\[
= .12 + .04 + .06 = .22
\]

6 pts c) Evaluate $EXY$.

\[
EXY = 0 (.06 + .06 + .7 + .11) + 1 (.06) + 2 (.04 + .08)
\]

\[
+ 3 (.11) + 4 (.09 + .10) + 6 (.12) + 8 (.10)
\]

\[
= .06 + .24 + .33 + .76 + .72 + .80
\]

\[
= 2.91
\]
d) Are the events "\( X = 3 \)" and "\( Y = 2 \)" independent events? Give convincing evidence in support of your "yes" or "no" answer.

\[
P(X = 3 \text{ and } Y = 2) = 0.12 \\
P(X = 3) \cdot P(Y = 2) = (0.3)(0.4) = 0.12
\]

Circle one: Yes  No

e) Are the random variables \( X \) and \( Y \) independent random variables? Give convincing evidence in support of your "yes" or "no" answer.

\[
P(X = 4 \text{ and } Y = 2) = 0.10 \\
P(X = 4) \cdot P(Y = 2) = (0.3)(0.4) = 0.12
\]

It is thus not the case that every \( f(x, y) \) is the product of marginals \( f(x)h(y) \).

Circle one: Yes  No

2. Account passwords at an online shopping site are 6 characters long. The elements of a password must be 1) lower case letters a,b,…,z (there are 26 of these), 2) digits 0,1,…,9 (there are 10 of these), or 3) one of the 6 "special characters @,#,$,%,&,* . A further requirement is that a password must contain at least one character of each of the 3 types. How many legal passwords are possible? You need NOT simplify your answer. (Hint: It might help to start by considering how many passwords there could be without the restriction.)

\[
\#	ext{without restriction} = (26+10+6)^6 \\
\#	ext{with only letters} = (26)^6 \\
\#	ext{with only digits} = 10^6 \\
\#	ext{with only special characters} = 6^6 \\
\#	ext{with only letters and digits} = (26+10)^6 - (26)^6 - (10)^6 \\
\#	ext{with only letters and special characters} = (26+6)^6 - (26)^6 - 6^6 \\
\#	ext{with only digits and special characters} = (10+6)^6 - (10)^6 - 6^6
\]

So \#	ext{legal} = (26+10+6)^6 - (26)^6 - (10)^6 - 6^6 - [(26+10)^6 - (26)^6 - (10)^6] - [(26+6)^6 - (26)^6 - 6^6] - [(10+6)^6 - (10)^6 - 6^6]
3. In the planning of a large engineering project, the number of days required to complete one step in the project is modeled as a continuous random variable with cumulative distribution function

\[
F(x) = \begin{cases} 
0 & \text{if } x < 100 \\
\frac{(x-100)^3}{(100)^3} & \text{if } 100 \leq x \leq 200 \\
1 & \text{if } x > 200 
\end{cases}
\]

a) Evaluate the probability that the time required to complete this step exceeds a target of 150 days.

\[
P(X > 150) = 1 - F(150) = 1 - \frac{(150-100)^3}{(100)^3} = 1 - \left(\frac{50}{100}\right)^3 = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}
\]

b) Find the pdf of the number of days needed to complete this step, \( f(x) \). (Be sure to say where the pdf is 0 and where any formula you produce is relevant.)

\[
f(x) = \frac{d}{dx} F(x) = \begin{cases} 
0 & \text{if } x < 100 \text{ or } x > 200 \\
3 \frac{(x-100)^2}{(100)^3} & \text{if } 100 < x < 200 
\end{cases}
\]

c) There is no loss to the project as long as this particular step requires no more than 150 days to complete. However, each additional day (beyond 150) required results in a $2000 loss. Set up completely but do not evaluate a definite integral giving the expected loss produced by any delay in this step of the project. (If you can't do b, you may write \( f(x) \) where appropriate instead of an explicit formula, but limits of integration must be correct numbers and all else must be complete.)

\[
h(x) = \text{delay loss} = \begin{cases} 
0 & \text{if } x < 150 \\
2000(x-150) & \text{if } x > 150 
\end{cases}
\]

We want \( E[h(X)] = \int h(x) f(x) \, dx \)

\[
= \int_{150}^{200} 2000(x-150)^3 \frac{(x-100)^2}{(100)^3} \, dx
\]
4. A company fills bottles with a liquid. The machine currently used to do the filling produces actual fill-levels that are normal with a mean ($\mu$) that depends upon operator set-up and a standard deviation ($\sigma$) of 3 ml. Currently, in order to make sure that few bottles have less than a label amount of liquid, the company sets $\mu$ at the label amount plus 5 ml.

7 pts  a) What fraction of bottles presently contain less than the label amount of liquid?

\[
Z = \frac{\text{label} - (\text{label} + 5)}{3} = -1.67
\]
\[P(Z < -1.67) = 0.0475\]

6 pts  b) Let $Y$ be the number of bottles among the next 10 filled that have receive at least the label amount of liquid. Use your answer a) and evaluate the probability that $Y$ is at least 9. (If you could not do a) you may use the incorrect figure of 20% here instead.)

\[Y = \mu \text{ with at least the label amount}\]
\[\text{Bi}(10, 1 - \cdot 0.0475)\]
\[P(Y \geq 9) = f(9) + f(10) = \binom{10}{9}(0.9525)^9(0.0475)^1(0.9525)^{10}\]

7 pts  c) A new filling machine can be purchased and installed for $100,000 and has a standard deviation of fill level ($\sigma$) that is 2 ml. Every 1 ml reduction in average fill level that the company makes produces a $50,000 per year savings in product not "given away." If this new machine is purchased and set up to give the same fraction of under-filled bottles as the present machine, how many years will be required to pay back investment in the machine?

\[
\frac{\text{label} - (\text{label} + ?)}{2} = -1.67
\]
\[-\ ? = -2(1.67)\]
\[? = 3.33\]

Thus $5 - 3.33 = 1.67$ ml reduction can be made in the mean fill level. This produces a $1.67(50) = 83.3k$ savings per year and thus $\frac{100k}{83.3k/\text{year}} = 1.2$ years are required for payback.
5. Suppose $X_1, X_2,$ and $X_3$ are independent Poisson random variables, each with mean $\lambda = 1$.

10 pts a) Use the laws of expectation and variance to find the mean of $Y = (X_1 + X_2)(X_2 + X_3)$

\[
E(Y) = E(X_1X_2 + X_1X_3 + X_2^2 + X_2X_3) = E(X_1X_2) + E(X_1X_3) + E(X_2X_3) + E(X_2^2)
\]

Since $X_1, X_2, X_3$ are independent,

\[
E(X_1X_2 + X_1X_3 + X_2X_3) = 1(1) + 1(1) + 1(1) + (1 + 1^2) = 5
\]

\[
\text{Var}(Y) = \frac{\text{Var}(X_1X_2) + \text{Var}(X_1X_3) + \text{Var}(X_2X_3) + \text{Var}(X_2^2)}{1 + 1 + 1 + 1^2} = \frac{5}{5} = 1
\]

b) Use the propagation of error formula to approximate the variance of the variable $Z = \frac{1 + X_1}{1 + X_1 + X_2}$

\[
\text{Var}(Z) \approx \left( \frac{\partial z}{\partial x_1} \right)^2 \text{Var}(X_1) + \left( \frac{\partial z}{\partial x_2} \right)^2 \text{Var}(X_2)
\]

\[
= \left( \frac{1 + x_1 + x_2}{(1 + x_1 + x_2)^2} - \frac{1 + x_1}{(1 + x_1 + x_2)^2} \right)^2 (1) + \left( -\frac{1 + x_1}{(1 + x_1 + x_2)^2} \right)^2 (1)
\]

\[
= \left( \frac{1}{9} \right)^2 (1) + \left( -\frac{2}{9} \right)^2 (1) = \frac{5}{81} = .0617
\]

\[
\text{Var}(Z) \approx .0617
\]