I have neither given nor received unauthorized assistance on this exam.

KEY

Name Signed  Date

Name Printed
1. A not-so-excellent marksman fires at a solid-circular target of radius 1 ft, and shots hitting the target might be modeled as uniformly distributed inside the circle. If one thinks of the center of the target as the origin of coordinate axes, and \((X, Y)\) as the coordinates of the random location of a hit on the target, a sensible joint pdf for \(X\) and \(Y\) is

\[
f(x, y) = \begin{cases} 
\frac{1}{\pi} & \text{if } x^2 + y^2 < 1 \\
0 & \text{otherwise}
\end{cases}
\]

and this is pictured to the right.

### a) For \(0 < r < 1\), what is the probability that a shot hitting the target lands within \(r\) feet of the center? (This is \(P\left[\sqrt{X^2 + Y^2} \leq r\right]\).)

The desired probability is

\[
\int \int_{\sqrt{x^2 + y^2} \leq r} \frac{1}{\pi} \, dx \, dy = \frac{\pi r^2}{\pi} = r^2
\]

### b) Find the marginal probability density for \(X\), \(g(x)\). (Be sure to say completely where any expression you propose for \(g(x)\) applies.)

This is \(\int f(x, y) \, dy\). For \(-1 < x < 1\) this is

\[
\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \, dy = 2 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \, dy = 2 \frac{\sqrt{1-x^2}}{\pi}
\]

So \(g(x) = \begin{cases} 
\frac{2}{\pi} \sqrt{1-x^2} & -1 < x < 1 \\
0 & \text{otherwise}
\end{cases}\)

### c) Find the conditional probability density for \(Y\) given that \(X = \frac{1}{2}\). (Again be very careful to say completely where any expression you propose for \(h\left(y \mid \frac{1}{2}\right)\) applies.)

This is \(f\left(\frac{1}{2}, y\right) / g\left(\frac{1}{2}\right)\) i.e.

\[
h\left(y \mid \frac{1}{2}\right) = \begin{cases} 
\frac{1}{2 \sqrt{1-(x)^2}} & -\frac{\sqrt{3}}{2} < y < \frac{\sqrt{3}}{2} \\
0 & \text{otherwise}
\end{cases}
\]

7 pts 7 pts 7 pts
2. A lathe turns out cylinders with diameters that are normally distributed with mean 3.02 cm and standard deviation .01 cm. (Model successive diameters as independent.) Engineering specifications/requirements are that a cylinder diameter be between 2.97 cm and 3.03 cm.

a) What is the probability that the next diameter is outside engineering specifications?

\[ P(\text{next is outside}) = 1 - P(2.97 < X < 3.03) \]

\[ z_1 = \frac{3.03 - 3.02}{.01} = 1 \]

\[ z_2 = \frac{2.97 - 3.02}{.01} = -5 \]

\[ P(-5 < z < 1) = \Phi(1) = .8413 \]

\[ P(\text{next is outside}) = 1 - .8413 = .1587 \]

If you were unable to do part a), you may use the incorrect value .21 in parts b) and c).

b) Based on your answer to part a), what is the mean number of cylinders that will be produced in order to find a first one that is out of engineering specifications?

\[ Y = \text{# required to find a first outside specifications} \]

\[ Y \sim \text{Geo}(.1587) \]

\[ EY = \frac{1}{.1587} \approx 6.30 \]

c) What is the probability that among the next 10 cylinders produced, exactly 2 have diameters that are out of specifications? (You do not need to simplify to a decimal number.)

\[ W = \text{# among the next 10 outside specifications} \]

\[ W \sim \text{Bi}(10, .1587) \]

\[ P(W = 2) = f(2) = \binom{10}{2} (.1587)^2 (.8413)^8 = .2844 \]
3. An injection molding machine produces "pre-forms" for plastic bottles. These are weighed with some measurement error. For the next pre-form produced, model

\[ W = \text{the actual weight of a produced part} \]

and

\[ U = \text{the error of weight measurement} \]

as independent random variables, and suppose that what is actually observed is

\[ M = W + U = \text{the measured weight} \]

Suppose further that \( E_W = .80 \text{ oz}, \sqrt{\text{Var}W} = .008 \text{ oz}, E_U = .01 \text{ oz}, \) and \( \sqrt{\text{Var}U} = .005 \text{ oz} \).

**a)** What are the values of \( EM \) and \( \sqrt{\text{Var}M} \)?

\[
EM = E(W+U) = EW + EU = .80 + .01 = .81
\]

\[
\text{Var} M = \text{Var}(W+U) \leq \text{Var} W + \text{Var} U = (.008)^2 + (.005)^2
\]

\[
\sqrt{\text{Var} M} = \sqrt{(.008)^2 + (.005)^2} = .0094 \text{ oz}
\]

**b)** Find the correlation between \( W \) and \( M \).

\[
\text{Cov}(W,M) = E(WM) - EWEM
\]

\[
= E(W(W+U)) - EW(EW+EU)
\]

\[
= EW^2 + EUW - EW^2 - EUW
\]

\[
= EW^2 - EW^2 + EUW - EUW
\]

\[
= \text{Var} W
\]

\[
= (.008)^2
\]

So \( \rho = \frac{(.008)^2}{(.008)(.0094)} = \frac{.008}{.0094} \approx .85 \)
4. A test for the presence of an unsafe level of Benzene contamination is not perfect. 70% of specimens with Benzene level exactly at the safety cut-off concentration are judged by the method to be unsafe, 10% are judged to be safe, and 20% produce inconclusive results when tested. On the other hand, while "blank" specimens (ones having only "background" levels of Benzene) are never judged to be unsafe, 40% of them produce inconclusive results, and only 60% of blanks are judged to be safe.

In a series of tests, 50% of specimens submitted are blanks and 50% have Benzene levels at exactly the cut-off concentration. One of these tests is selected at random.

5 pts  

a) Name two mutually exclusive events in this context.

"blank" and "judged unsafe" are mutually exclusive

5 pts  

b) What is the probability that the test is inconclusive?

\[
P(\text{inconclusive}) = P(\text{inconclusive} | \text{blank}) P(\text{blank}) + P(\text{inconclusive} | \text{at cut-off level}) P(\text{at cut-off level})
\]

\[
= (.4)(.5) + (.2)(.5)
\]

\[
= .3
\]

5 pts  

c) Given that the test is inconclusive, what is the (conditional) probability that the specimen is a blank?

\[
P(\text{blank} | \text{inconclusive}) = \frac{P(\text{blank and inconclusive})}{P(\text{inconclusive})} = \frac{P(\text{inconclusive} | \text{blank}) P(\text{blank})}{P(\text{inconclusive})}
\]

\[
= \frac{(0.4)(0.5)}{0.3} = \frac{2}{3}
\]

5 pts  

d) Are the events "inconclusive result" and "blank specimen" independent events? Explain!

No they are not. \( P(\text{blank} | \text{inconclusive}) = \frac{2}{3} \neq .5 = P(\text{blank}) \)
A crafty statistics professor offers you your choice of 2 deals. You may observe the value of either

\( X \sim \text{Exponential}(1) \)

or

\( Y \sim \text{Poisson}(1) \)

(you must choose before seeing the value of the random variable). He will then reward you with a payoff that is

- 0 if the value you observe is less than .5
- the value you observe if it is between .5 and 2.5
- 1 if the value you observe is bigger than 2.5

Which deal should you take to maximize your expected payoff? (You will get no credit for a correct guess unsupported by appropriate calculations.)

Compute and compare expected payoffs for the 2 deals.

**Exponential Deal:** Here the expected payoff is

\[
E_{\text{payoff}} = \int_{0}^{\infty} 0(e^{-x}) \, dx + \int_{.5}^{2.5} x(e^{-x}) \, dx + \int_{2.5}^{\infty} (e^{-x}) \, dx
\]

\[
= -xe^{-x} \bigg|_{.5}^{2.5} - x(e^{-x}) \bigg|_{.5}^{2.5} + (e^{-x}) \bigg|_{2.5}^{\infty}
\]

\[
= 1.5e^{-0.5} - 2.5e^{-2.5} + e^{-0.5} = 1.5e^{-0.5} - 2.5e^{-2.5} = .7046
\]

**Poisson Deal:** Here the expected payoff is

\[
E_{\text{payoff}} = 0f(0) + 1f(1) + 2f(2) + 1(1-(f(0)+f(1)+f(2))
\]

\[
= 1 - f(0) + f(2)
\]

\[
= 1 - e^{-1} + \frac{e^{-1}(1)^2}{2}
\]

\[
= 1 - .5e^{-1} = .8161
\]

The Poisson deal is better for you (at least in terms of \( E(\text{payoff}) \)).