Statistics and Measurement

(Section 2.2 of Vardeman and Jobe plus some)
A Measurement System is One’s “Looking Glass”
Basic Issues in Metrology

• **Validity** (Am I really tracking what I want to track?)

• **Precision** (Consistency of measurement)

• **Accuracy** (Getting the “right” answer on average)
Basic Measurement Model (that allows for the reality of “error”)

• For a single item:

\[ y = x + \varepsilon \]

where:

\( x = \) the "true" value
\( y = \) the measured value

and

\( \varepsilon = \) the (random) measurement error (a r.v.)

with mean \( \beta \) and std. dev. \( \sigma_{\text{measurement}} \)
Measurement Model

\[
y = x + b + \sigma_{\text{measurement}}
\]
Measurement Model

• Multiple items

\[ y = x + \varepsilon \]

where now \( x \) is random with mean \( \mu_x \) and standard deviation \( \sigma_x \) (this measures “real” process or item-to-item variation)

Now

\[ \mu_y = \mu_x + \beta \quad \text{and} \quad \sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{measurement}}^2} > \sigma_x \]
Estimating Measurement Variation

• For a sample of $m$ measurements on the same item producing $ar{y}$ and $s$
  – the mean of $\bar{y}$ is $x + \beta$
  – the mean of $s^2$ is $\sigma^2_{\text{measurement}}$
  – the interval

$$\left( \sqrt{\frac{(m-1)s^2}{\chi^2_{m-1,\text{upper}}}}, \sqrt{\frac{(m-1)s^2}{\chi^2_{m-1,\text{lower}}}} \right)$$

estimates $\sigma_{\text{measurement}}$
Estimating Part/Process Variation

• For a sample of measurements on $n$ different items producing $\bar{y}$ and $s_y$
  – the mean of $y$ is $\mu_x + \beta$
  – the mean of $s_y^2$ is $\sigma_x^2 + \sigma_{\text{measurement}}^2$ (not $\sigma_x^2$)
  – the interval

\[
\left( \sqrt{\frac{(n-1)s_y^2}{\chi_{n-1,\text{upper}}^2}}, \sqrt{\frac{(n-1)s_y^2}{\chi_{n-1,\text{lower}}^2}} \right)
\]

estimates

\[
\sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{measurement}}^2}
\]
Estimating Part/Process Variation

• An estimate of unit-to-unit variation free of measurement variation is (for \(m\) measurements on a single unit and \(n\) on different units) is

\[
\hat{\sigma}_x = \sqrt{\max\left(0, s_y^2 - s^2\right)}
\]

and this estimate has standard error

\[
\sqrt{\frac{1}{2} \left( \frac{1}{s_y^2 - s^2} \right) \left( \frac{s_y^4}{n - 1} + \frac{s^4}{m - 1} \right)}
\]
Example

- \( n=30 \) different widget diameters with \( \bar{y} = 1.002 \) inch and \( s_y = .05 \) inch
- \( m=10 \) measurements made on the same widget diameter produce \( \bar{y} = .997 \) inch and \( s = .008 \) inch

- Estimate part-to-part standard deviation as \( \hat{\sigma}_x = \sqrt{\text{max} \left( 0, (0.05)^2 - (0.008)^2 \right)} = .049 \) inch

with standard error \( \sqrt{\frac{1}{2} \left( \frac{1}{(0.05)^2 - (0.008)^2} \right) \left( \frac{(0.05)^4}{30-1} + \frac{(0.008)^4}{10-1} \right)} = .007 \) inch
Improving Accuracy Through Calibration and Curve Fitting

• Calibration experiments get “true”/gold-standard-measurement values $x$ and “local” measurements $y$ and seek a “conversion” method

• The relevant statistical methodology is curve-fitting/regression analysis

• Regression analysis can provide both “point conversions” and measures of uncertainty (the latter through inversion of “prediction limits”)
Simple Linear Regression

- SLR Model is

\[ y = \beta_0 + \beta_1 x + \epsilon \]

- Pictorially:
Mandel (NBS/NIST) Example

- “Gold-standard” and “local” measurements on \( n = 14 \) specimens (units not given)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>647</td>
<td>605</td>
<td>1594</td>
<td>1470</td>
</tr>
<tr>
<td>728</td>
<td>675</td>
<td>1896</td>
<td>1762</td>
</tr>
<tr>
<td>1039</td>
<td>965</td>
<td>1983</td>
<td>1739</td>
</tr>
<tr>
<td>1095</td>
<td>995</td>
<td>2136</td>
<td>1918</td>
</tr>
<tr>
<td>1116</td>
<td>1018</td>
<td>2192</td>
<td>1983</td>
</tr>
<tr>
<td>1194</td>
<td>1117</td>
<td>2224</td>
<td>2008</td>
</tr>
<tr>
<td>1557</td>
<td>1422</td>
<td>2244</td>
<td>2010</td>
</tr>
</tbody>
</table>
Mandel Example

\[ y = 42.2163 + 0.881819 \times \]

\[ S = 25.3258 \quad R-Sq = 0.998 \]

95% Prediction Limits

Estimated \( x \) if \( y = 1500 \)

\( (x=1653) \)

95\% Interval for \( x \) if \( y=1500 \)
Evaluating Measurement Precision in an “R&R” Study

• Often there are “operator effects” that should be considered part of measurement imprecision

• “Repeatability” variation is variation characteristic of one operator remeasuring one part

• “Reproducibility” variation is variation characteristic of many operators measuring a single part once (exclusive of repeatability variation)
Typical Gauge R&R Data Layout

- E.g. $I=2$ parts, $J=3$ operators and $m=2$ repeats per “cell”

<table>
<thead>
<tr>
<th>Part</th>
<th>Operator</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$y_{111}$</td>
<td>$y_{121}$</td>
<td>$y_{131}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{112}$</td>
<td>$y_{122}$</td>
<td>$y_{132}$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$y_{211}$</td>
<td>$y_{221}$</td>
<td>$y_{231}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_{212}$</td>
<td>$y_{222}$</td>
<td>$y_{232}$</td>
</tr>
</tbody>
</table>
Estimates of R&R “Sigmas”

• Repeatability variation is “within cells”

\[ \hat{\sigma}_{\text{repeatability}} = \frac{\bar{R}}{d_2(m)} \]  
(based on ranges)

or

\[ \hat{\sigma}_{\text{repeatability}} = \sqrt{MSE} \]  
(based on ANOVA)

• Describing reproducibility variation requires more subtlety (common range-based methods are wrong)
Estimates of Reproducibility Sigma (Using Ranges)

- One must correct/allow for repeatability variation in considering differences between operators (as earlier in $\hat{\sigma}_x$)
  - Let $\Delta_i$ be the range of the part $i$ cell means
  - Further, let $\bar{\Delta}$ be the mean of these ranges

$$\hat{\sigma}_{\text{reproducibility}} = \sqrt{\max \left( 0, \left( \frac{\bar{\Delta}}{d_2(J)} \right)^2 - \frac{1}{m} \left( \frac{\bar{R}}{d_2(m)} \right)^2 \right)}$$

- ANOVA-based estimator is on page 27 of V&J$_{18}$
BTW ... Standard Errors

- For estimates of repeatability sigma:
  \[ \frac{d_3(m) \cdot \sigma_{\text{repeatability}}}{d_2(m) \sqrt{IJ}} \]

  based on ranges

  and

  \[ \frac{\hat{\sigma}_{\text{repeatability}}}{\sqrt{2IJ(m-1)}} \]

  based on ANOVA

- For ANOVA estimate of reproducibility sigma use Chiang’s program (take the Stat531 link from Vardeman’s Web page)

  … “usually” these are HUGE
I=3, J=3 and m=2 Example

<table>
<thead>
<tr>
<th>Part</th>
<th>Operator 1</th>
<th>Operator 2</th>
<th>Operator 3</th>
<th>( \Delta_1 )</th>
<th>( \Delta_2 )</th>
<th>( \Delta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>mean = .34730</td>
<td>mean = .34660</td>
<td>mean = .34715</td>
<td>.00070</td>
<td>.00065</td>
<td>.00065</td>
</tr>
<tr>
<td></td>
<td>range = 0</td>
<td>range = .0002</td>
<td>range = .0001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>mean = .34710</td>
<td>mean = .34645</td>
<td>mean = .34710</td>
<td></td>
<td>.00065</td>
<td></td>
</tr>
<tr>
<td></td>
<td>range = 0</td>
<td>range = .0001</td>
<td>range = 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>mean = .34720</td>
<td>mean = .34655</td>
<td>mean = .34710</td>
<td></td>
<td></td>
<td>.00065</td>
</tr>
<tr>
<td></td>
<td>range = 0</td>
<td>range = .0003</td>
<td>range = 0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

so \( \bar{R} = .0007/9 = .000078 \) and \( \bar{\Delta} = .00067 \)

\( \hat{\sigma}_{\text{repeatability}} = \frac{\bar{R}}{d_2(2)} = \frac{.000078}{1.128} = .000069 \) inch

(standard error for this is \( \frac{.853}{1.128\sqrt{3.3}} \) times this value)

\( \hat{\sigma}_{\text{reproducibility}} = \sqrt{\max\left(0, \left(\frac{.00067}{1.693}\right)^2 - \left(\frac{.000069}{2}\right)^2\right)} = .000391 \) inch
Workshop Exercises

• See problem 5.8, page 249 and the $n=8$ measured hardnesses there. Suppose that previous measurement of a single part $m=5$ times gave $s=.044$ mm. Find $\sigma_x$ and a standard error for this estimate.

• For the Mandel example, about what “corrected” value would you associate with a local measurement of 1000? What 95% limits would you associate with this?