CUSUM Charts

(Section 4.2 of Vardeman and Jobe)

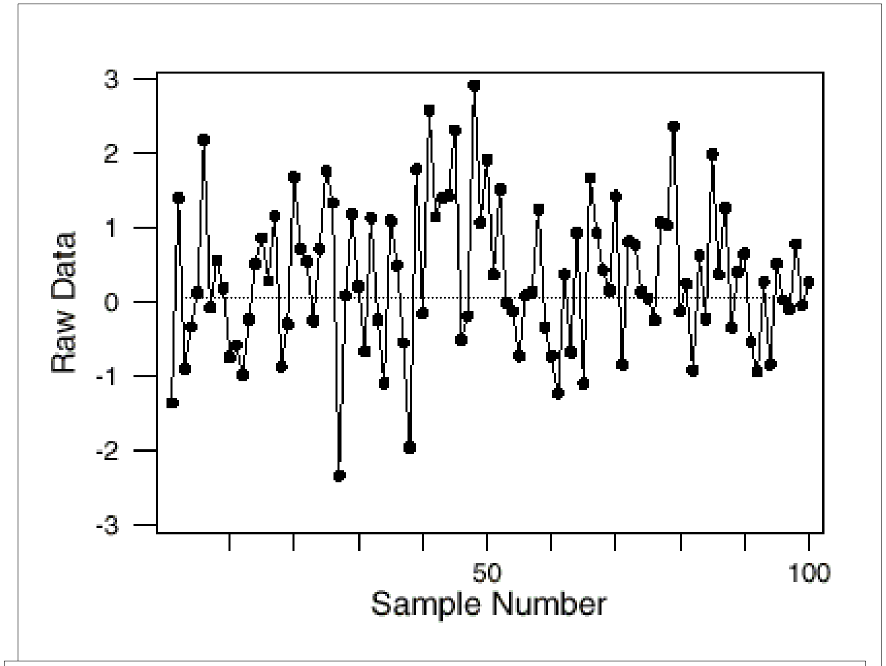
Basic Motivation

- It may be difficult to see small changes in the distribution of a plotted statistic Q on a Shewhart chart
- It can be easier to see those changes if one "accumulates" deviations from some standard value for Q
- A "raw" Cumulative Sum sequence is defined by

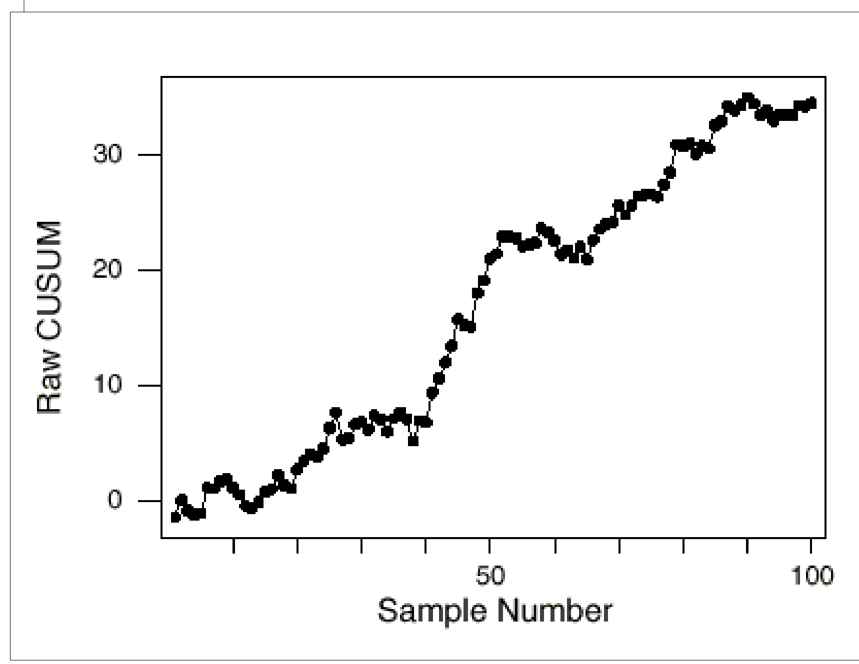
$$CUSUM_i = (Q_i - k) + CUSUM_{i-1}$$

Example

- Qs Normal with mean $.5$ and std dev 1.0

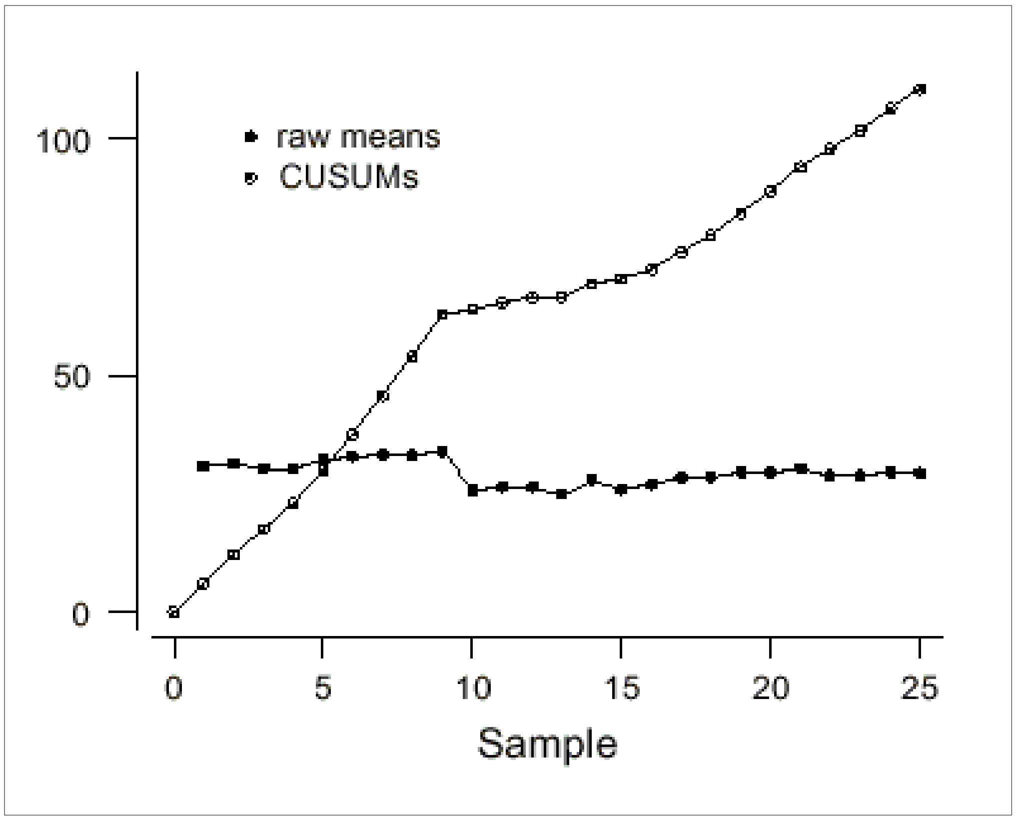


- $CUSUM$s of Qs with $CUSUM_0 = 0$ and $k = 0$



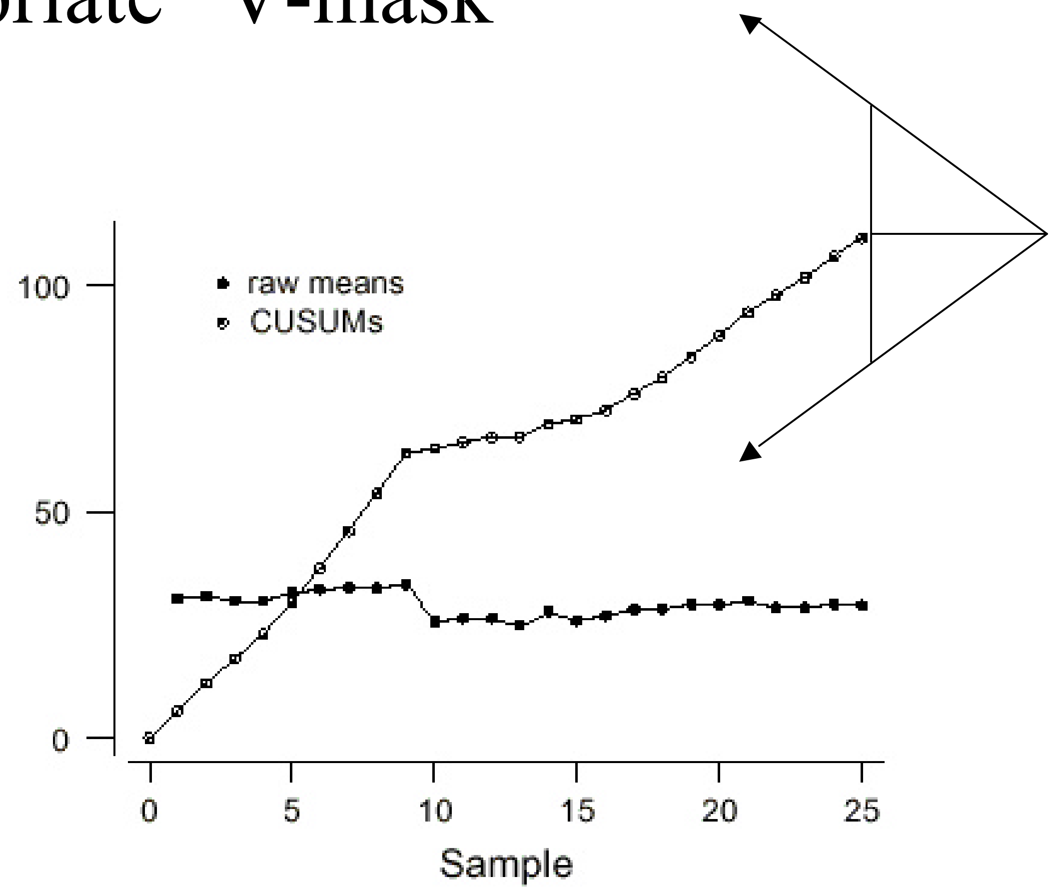
Interpreting a CUSUM

- It is the *slope* on a CUSUM plot that indicates how average Q differs from k
- Example 4.2 (k=*25*)



How to Use This in SPM?

- One possibility (I believe, essentially never used in real practice!) is to apply an appropriate "V-mask"



One-Sided "Decision Interval" CUSUMs

- "High Side" CUSUM

$$U_i = \max\left[0, (Q_i - k_1) + U_{i-1}\right]$$

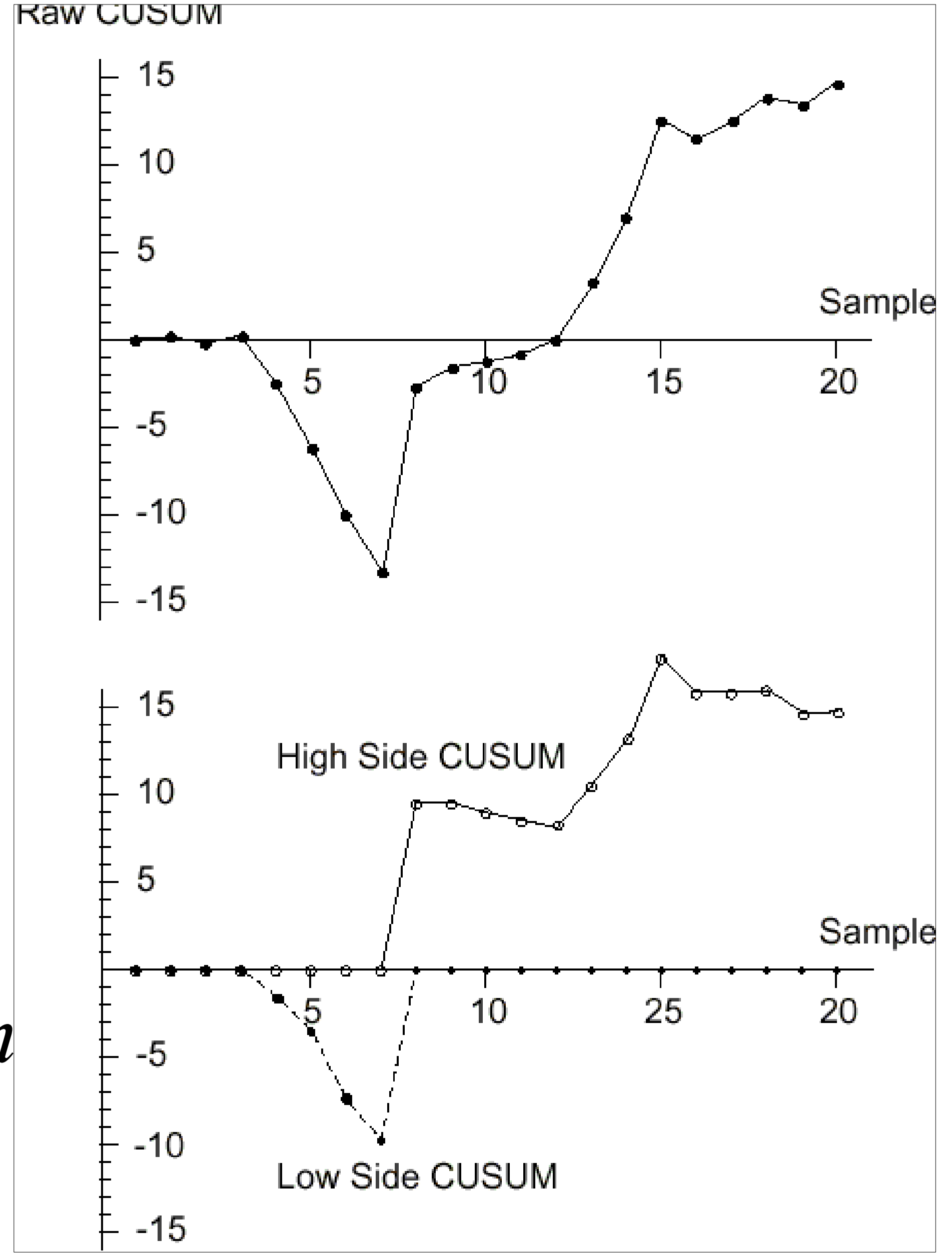
- "Low Side" CUSUM

$$L_i = \min\left[0, (Q_i - k_2) + L_{i-1}\right]$$

- All of k_1 and/or k_2, U_0 and/or L_0 are (appropriately chosen) parameters of the scheme

More Decision Interval CUSUMs

- Raw CUSUMs with a "restart" feature
- "Reference values" k_1 and k_2 are typically chosen above and below an ideal Q
- "Out-of-control" signals derive from decision levels h and $-h$



Choice of Scheme Parameters

- Common choice of starting values is

$$U_0 = 0 \text{ and/or } L_0 = 0$$

- For a given choice of k_1 and/or k_2 and a normal "all-OK" distribution for Q, Tables 4.5 and 4.6 can be used to pick h providing a desired mean time between false-alarms (a desired "all-OK ARL")
  - To enter the tables one must standardize by

$$K = \frac{k_1 - m_Q}{s_Q} \quad \text{or} \quad K = \frac{m_Q - k_2}{s_Q}$$

and read out H

More Choice of Parameters

– Then set $h = H s_Q$
– For simultaneous high and low side schemes with

$$\frac{k_1 + k_2}{2} = m_Q$$

and *all-OK ARL=370*, Table 4.6 shows appropriate choice of H to be

K					
.25	.50	.75	1.00	1.25	1.50
8.01	4.77	3.34	2.52	1.99	1.60

More Choice of Parameters

- "Optimal" choice of k_1 and/or k_2 is possible (for given all-OK ARL and potential shift in mean Q)
  - For detecting a shift in mean Q of size $\boldsymbol{d}$, approximately optimal reference values are

$$k_1^{\text{opt}} = m_Q + \frac{\boldsymbol{d}}{2} \quad \text{and/or} \quad k_2^{\text{opt}} = m_Q - \frac{\boldsymbol{d}}{2}$$

Example (4.3)

- Process monitoring with
  - $Q = \overline{x}$ based on $n = 4$
  - all-OK process parameters $m = 9.0$ and $s = 1.6$ (so the all-OK distribution of $Q = \overline{x}$ has $m_Q = 9.0$ and $s_Q = s / \sqrt{n} = 1.6 / \sqrt{4} = .8$)
  - *all-OK ARL=370* desired
  - Quickest possible detection of a change of size *2.0* in the process mean (and therefore in mean *Q*) desired
- Set-up of combination high- and low-side scheme?

Example Continued

- 1st choose the reference values

$$k_1 = 9.0 + \frac{2.0}{2} = 10.0 \text{ and } k_2 = 9.0 - \frac{2.0}{2} = 8.0$$

- Next choose *h*
  - $K = \frac{k_1 - m_Q}{s_Q} = \frac{10.0 - 9.0}{.8} = 1.25$
  - From Table 4.6 (or slide 9), *H=1.99*
  - So take *h=(1.99)(.8)=1.592*
- Use starting values $U_0 = 0$ and $L_0 = 0$

Predicted Behavior?

- For given k_1 and/or k_2 and h , with $U_0 = 0$ and/or $L_0 = 0$ and normal Q, it is possible to find (not-all-OK) ARLs (mean times to detection)
  - Use formula (4.15) and
    - Formula (4.16) or (4.17) to find parameters needed to enter Table A.4 for one-sided schemes (one inputs the mean and standard deviation of Q)
    - Formulas (4.18) and (4.19) to find parameters needed to enter Table A.5 for combined high- and low-side schemes

Perspective

- Not a tool for plotting "by hand"
- CUSUM and EWMA charts have essentially the same ARLs under small departures from standard conditions, *supposing those departures pertain from time 1 on*
- CUSUM charts do NOT have the have poor "worst case" behavior of EWMA schemes! (one is never "any further from an alarm" than at time 1)

Possible "Enhancements" (see Section 4.2.2)

- "Fast Initial Response" CUSUMs

$$U_0 = \frac{h}{2} \text{ and/or } L_0 = \frac{-h}{2}$$

- Shewhart/CUSUM Combinations
  - CUSUM plus a "3.5 sigma" or "4.0 sigma" Shewhart chart "combines the best of both worlds"

Workshop Exercises

- Find $k = 1.0$ raw *CUSUM*s for the *Q*s below

i	Q_i	$Q_i - k$	$CUSUM_i$
0			0
1	1		
2	-3		
3	0		
4	1		
5	20		
6	-5		
7	0		
8	1		

Workshop Exercises

- Find $k_1 = 1.0$ and $k_2 = -1.0$ $CUSUM$s

i	Q_i	$Q_i - k_1$	U_i	$Q_i - k_2$	L_i
0			0		0
1	1				
2	-3				
3	0				
4	1				
5	20				
6	-5				
7	0				
8	1				

Workshop Exercises

- Suppose $Q = x$ and process standards are

  $$m = 0 \text{ and } s = 1$$

  If quick detection of a change in process mean of size $d = .5$ is of importance, and *all-OK ARL=370*, set up a combined high- and low-side CUSUM scheme

  (what are appropriate k_1, k_2 and h ?)