Module 8

Some multi-sample examples

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Example 7

We finish up with a couple of interesting multi-sample examples of Bayes analyses. The first is a "one way random effects analysis" example.

In a 2006 *Quality Engineering* paper "Calibration, Error Analysis, and Ongoing Measurement Process Monitoring for Mass Spectrometry," Vardeman, Wendelberger, and Wang discuss a Bayes analysis of 44 measurement of a spectrometer's sensitivity to Argon gas, made across 3 days. With

$$DS_{tj} = \text{the device sensitivity computed from specimen } j \text{ on day } t$$

they used a decomposition

$$DS_{tj} = \mu_{\text{S}} + \delta_t + \epsilon_{tj}$$

for μ_{S} a fixed unknown "true" device sensitivity, δ_t a random "day t" deviation in sensitivity, and ϵ_{tj} a random specimen deviation.

Example 7 (Data)

Here are the data for the study (device sensitivities, where units are $\text{mol}/\text{A s}$).

Day 1		Day 2		Day 3	
31.3	27.8	32.5	30.5	31.7	28.3
31.0	28.2	32.2	28.4	29.8	28.3
29.4	28.4	31.9	28.5	29.6	28.3
29.2	28.7	30.2	28.8	29.0	29.2
29.0	29.7	30.2	28.8	28.8	29.7
28.8	30.8	29.5	30.6	29.6	31.1
28.8	30.1	30.8	31.0	28.9	
27.7	29.9				
27.7					

Example 7 (Data Model)

Assuming that the δ_t are independent draws from a normal distribution with mean 0 and standard deviation σ_δ, independent of the ϵ_{tj} that are independent random draws from a normal distribution with mean 0 and standard deviation σ, this is a problem with parameters $\mu_{\text{S}}, \sigma_\delta$, and σ.

The model can be rephrased as

$$\mu_t \equiv \mu_{\text{S}} + \delta_t = \text{the day } t \text{ sensitivity} \quad \sim \text{N}\left(\mu_{\text{S}}, \sigma_\delta^2\right) \text{ for } t = 1, 2, 3$$

and given the values of μ_t, the specimen sensitivities are

$$DS_{tj} = \mu_{\text{S}} + \delta_t + \epsilon_{tj} \quad \sim \text{N}\left(\mu_t, \sigma^2\right)$$

Example 7 (Prior)

We considered the specification of a prior for the parameters $\mu_{\mathrm{S}}, \sigma_{\delta}$, and σ,

$$
\begin{aligned}
\mu_{\mathrm{S}} &\sim \mathrm{N}\left(0, 10^6\right) \text{ independent of} \\
\frac{1}{\sigma_{\delta}^2} &\sim \mathrm{Gamma}\left(.001, .001\right) \text{ independent of} \\
\frac{1}{\sigma^2} &\sim \mathrm{Gamma}\left(.001, .001\right)
\end{aligned}
$$

These were intended to be relatively non-informative (but nevertheless proper) priors for the parameters.

The `WinBUGS` code used in the paper is in the file

`BayesASQEx7.odc`

and listed on the next two panels.

Example 7 (WinBUGS Code)

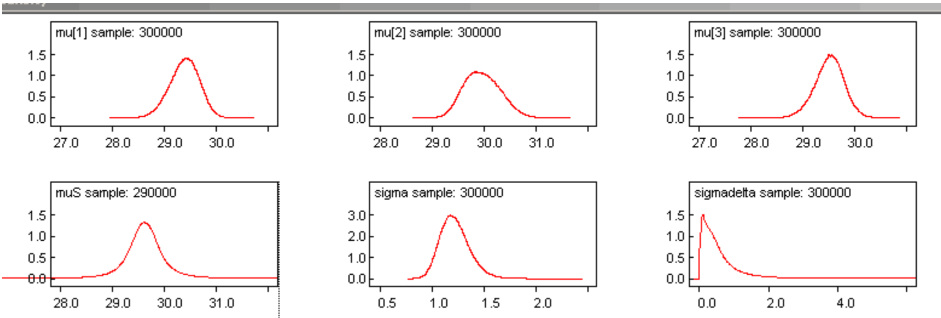
```
list(sens=c(31.3,31.0,29.4,29.2,29.0,28.8,28.8,27.7,
27.7,27.8,28.2,28.4,28.7,29.7,30.8,30.1,29.9,32.5,32.2,
31.9,30.2,30.2,29.5,30.8,30.5,28.4,28.5,28.8,28.8,30.6,
31.0,31.7,29.8,29.6,29.0,28.8,29.6,28.9,28.3,28.3,28.3,
29.2,29.7,31.1),ind=c(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
2,2,2,2,2,2,2,2,2,2,2,2,2,2,3,3,3,3,3,3,3,3,3,3,3,3,3),
N=44)

list(mu=c(3,3,3),tau=1,muS=0,taudelta=1)
```

Example 7 (WinBUGS Code cont.)

```
model {
 for(i in 1:N) {
 sens[i]~dnorm(mu[ind[i]],tau)
 }
 for(i in 1:3) {
 mu[i]~dnorm(muS,taudelta)
 }
 tau~dgamma(0.001,0.001)
 sigma<-1/sqrt(tau)
 muS~dnorm(0.0,1.0E-6)
 taudelta~dgamma(0.001,0.001)
 sigmadelta<-1/sqrt(taudelta)
}
```

The next figure shows some summaries from a `WinBUGS` session based on the code.

Example 7 (cont.)



Node statistics

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
mu[1]	29.38	0.284	0.001202	28.8	29.4	29.9	11001	300000
mu[2]	29.97	0.347	0.001801	29.35	29.95	30.68	11001	300000
mu[3]	29.51	0.2842	9.391E-4	28.91	29.52	30.04	11001	300000
muS	29.62	0.9781	0.002014	28.55	29.62	30.68	21001	290000
sigma	1.221	0.1397	3.262E-4	0.9827	1.209	1.528	11001	300000
sigmadelta	0.6353	1.825	0.007307	0.03671	0.3903	2.685	11001	300000

Figure: Summaries of a Bayes analysis of a spectrometer's sensitivity to Argon gas

Example 7 (cont.)

Notice that the mean(s) in this problem are far more precisely determined than are the standard deviations. That is a well known phenomenon. It takes a very large sample size to make definitive statements about variances or standard deviations ... and in the case of σ_δ, the appropriate "sample size" is 3! (Tests were made on only 3 days.)

Example 8

Two versions of an industrial process are run with the intention of comparing effectiveness. (There is an "old/#1" and a "new/#2" process.) Six different batches of raw material are used in the study. For

$$y_{ij} = \text{the yield of the } j\text{th run made using raw material batch } i \quad ,$$

process data are below.

i	j	process	y_{ij}	i	j	process	y_{ij}	i	j	process	y_{ij}
1	1	1	82.72	4	1	1	87.77	5	7	2	78.23
1	2	1	78.31	4	2	1	84.42	5	8	2	76.40
1	3	1	82.20	4	3	1	84.82	6	1	2	81.64
1	4	1	81.18	5	1	1	78.61	6	2	2	83.04
2	1	1	80.06	5	2	1	77.47	6	3	2	82.40
2	2	1	81.09	5	3	1	77.80	6	4	2	81.93
3	1	1	78.71	5	4	1	81.58	6	5	2	82.96
3	2	1	77.48	5	5	1	77.50				
3	3	1	76.06	5	6	2	78.73				

Example 8 (Data Model)

We'll consider an analysis based on the model

$$y_{ij} = \mu_{\text{process}(ij)} + \beta_i + \varepsilon_{ij}$$

where $\text{process}(ij)$ takes the value either 1 or 2 depending upon which version of the process is used, μ_1 and μ_2 are mean yields for the two versions of the process, the β_i are $\text{N}\left(0, \sigma_\beta^2\right)$ independent of the ε_{ij} which are $\text{N}\left(0, \sigma^2\right)$, and the parameters of the model are $\mu_1, \mu_2, \sigma_\beta,$ and σ.

This is (by the way) a so-called "mixed effects model." (The μ's are "fixed effects," of interest in their own right. The β's are "random effects" primarily of interest for what they tell us about σ_β, that measures random batch-to-batch variability.)

Example 8 (Priors)

What was presumably a fairly "non-informative" choice of prior distribution for the model parameters was

$$
\begin{aligned}
\mu_1 &\sim \text{N}\left(0, 10^6\right) \text{ independent of} \\
\mu_2 &\sim \text{N}\left(0, 10^6\right) \text{ independent of} \\
\ln\left(\sigma_\beta\right) &\sim \text{"Uniform } (-\infty, \infty)\text{"/"flat" independent of} \\
\ln\left(\sigma\right) &\sim \text{"Uniform } (-\infty, \infty)\text{"/"flat"}
\end{aligned}
$$

`WinBUGS` code for analysis of this situation is in the file

`BayesASQEx8.odc`

and listed on the next panel two panels.

Example 8 (WinBUGS Code)

```
model {
for (i in 1:2) {process[i]~dnorm(0,.000001)}
diff<-process[2]-process[1]
logsigb~dflat()
taubatch<-exp(-2*logsigb)
sigmabatch<-exp(logsigb)
for (j in 1:7) {batch[j]~dnorm(0,taubatch)}
logsig~dflat()
tau<-exp(-2*logsig)
sigma<- exp(logsig)
for (l in 1:27) {mu[l]<-process[p[l]]+batch[b[l]]}
for (l in 1:27) {y[l]~dnorm(mu[l],tau)}
}
```

Example 8 (WinBUGS Code cont.)

```
list(b=c(1,1,1,1,2,2,3,3,3,4,4,4,5,5,5,5,
5,5,5,5,6,6,6,6,6,7,7),
p=c(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,
2,2,2,2,2,2,1,2),
y=c(82.72,78.31,82.20,81.18,80.06,81.09,
78.71,77.48,76.06,87.77,84.42,84.82,78.61,
77.47,77.80,81.58,77.50,78.73,78.23,76.40,
81.64,83.04,82.40,81.93,82.96,NA,NA))
```

Note that the code calls for simulation of new responses from a 7th batch under both of the possible process conditions (the 26th and 27th y's). These come from posterior predictive distributions.

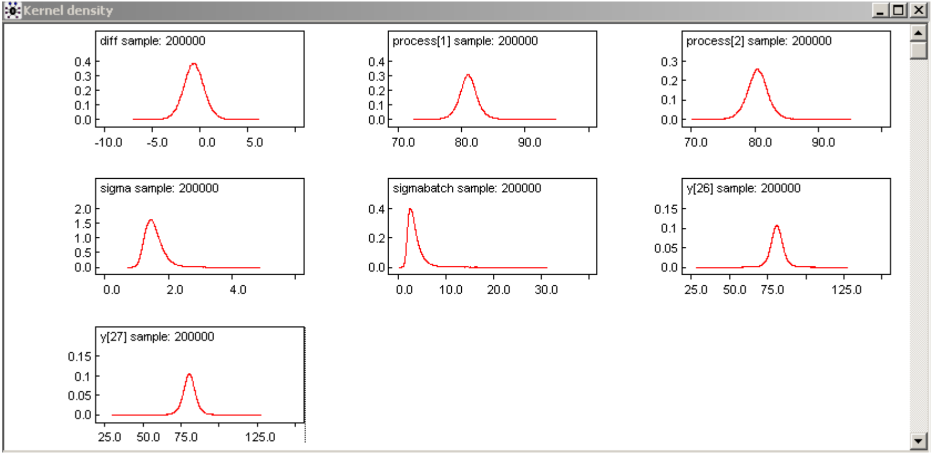
Example 8 (cont.)

Here is WinBUGS output that shows clearly that not enough has been learned from these data to say definitively how the processes compare.

Node statistics

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
batch[1]	0.01095	1.678	0.02521	-3.434	0.03386	3.276	7001	200000
batch[2]	-0.4584	1.77	0.02425	-4.081	-0.4266	2.965	7001	200000
batch[3]	-3.348	1.735	0.02495	-6.991	-3.278	-0.1003	7001	200000
batch[4]	4.165	1.729	0.02479	0.7914	4.142	7.657	7001	200000
batch[5]	-2.459	1.615	0.02603	-5.798	-2.421	0.6801	7001	200000
batch[6]	1.851	1.818	0.02661	-1.755	1.841	5.487	7001	200000
batch[7]	-0.007046	3.744	0.008558	-7.521	-0.01226	7.522	7001	200000
diff	-0.6555	1.082	0.005232	-2.794	-0.6595	1.497	7001	200000
process[1]	81.09	1.567	0.02558	78.03	81.06	84.34	7001	200000
process[2]	80.43	1.755	0.02696	76.96	80.42	83.96	7001	200000
sigma	1.541	0.2764	8.625E-4	1.112	1.504	2.187	7001	200000
sigmabatch	3.429	1.5	0.01263	1.646	3.091	7.238	7001	200000
y[26]	81.08	4.348	0.02703	72.52	81.04	89.84	7001	200000
y[27]	80.43	4.426	0.0282	71.65	80.42	89.28	7001	200000

Example 8 (cont.)



Example 9

As a final example of the power of Bayes analysis to handle what would otherwise be quite non-standard statistical problems, consider the following situation.

A response, y, has mean known to increase with a covariate/predictor, x, and is investigated in a study where (coded) values $x = 1, 2, 3, 4, 5, 6$ are used and there are 3 observations for each level of the predictor. Suppose the form of the dependence of the mean of y on x (say $\mathrm{E}y_{xj} = \mu_x$) is not something that we wish to specify beyond the restriction that

$$\mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \leq \mu_5 \leq \mu_6$$

(So, for example, simple linear regression is not an appropriate statistical methodology for analyzing the dependence of mean y on x.) We consider an analysis based on a model

$$y_{xj} = \mu_x + \varepsilon_{xj}$$

where the (otherwise completely unknown) means $\mu_1, \mu_2, \ldots, \mu_6$ satisfy the order restriction and the ε_{xj} are independent $\mathrm{N}\left(0, \sigma^2\right)$ variables.

Example 9 ("Data")

Here are some hypothetical data and summary statistics for this example.

x	y_{xj}'s	$\bar{y}_x$
1	0.9835899, −0.5087186, 1.0450089	.51
2	0.6815755, −2.1739497, 1.0464128	−.15
3	1.3717484, 1.1350734, 0.3384970	.95
4	6.5645035, 5.0648255, 6.0209295	5.88
5	6.5766160, 5.8730637, 7.4934093	6.65
6	7.8030626, 8.2207331, 6.7444797	7.59

and $\sqrt{MSE} = .99$.

Example 9 (Prior)

One way to make a simple `WinBUGS` analysis of this problem is to set

$$\delta_i = \mu_i - \mu_{i-1} \text{ for } i = 2, 3, \ldots, 6$$

and use priors

$$\begin{aligned} \mu_1 &\sim \text{N}\left(0, 10^4\right) \text{ independent of} \\ \delta_i &\sim \text{Uniform}\left(0, 10\right) \text{ for } i = 2, \ldots, 6 \text{ independent of} \\ \ln\left(\sigma\right) &\sim \text{"Uniform } \left(-\infty, \infty\right)\text{"/"flat"} \end{aligned}$$

`WinBUGS` code for analysis of this situation is in the file

`BayesASQEx9.odc`

and is listed on the next two panels.

Example 9 (WinBUGS Code)

```
model {
 mu1 ~dnorm(0,.0001)
 mu[1] <- mu1
 for (i in 2:6) {
 delta[i] ~dunif(0,10)}
 for (i in 2:6) {
 mu[i] <- mu[i-1]+delta[i]}
 logsigma ~dflat()
 sigma <- exp(logsigma)
 tau <- exp(-2*logsigma)
 for (j in 1:N) {
 y[j] ~dnorm(mu[group[j]],tau)}
}
```

Example 9 (WinBUGS Code cont.)

```
list(N=18,group=c(1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6),
y=c(0.9835899,-0.5087186,1.0450089,0.6815755,-2.1739497,
1.0464128,1.3717484,1.1350734,0.3384970,6.5645035,
5.0648255,6.0209295,6.5766160,5.8730637,7.4934093,
7.8030626,8.2207331,6.7444797))

list(mu1=0,logsigma=0)
```

Output on the next two panels shows how easy it is to get sensible inferences in this nonstandard problem from a `WinBUGS` Bayes analysis.

Example 9 (cont.)

Node statistics

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
delta[2]	0.4471	0.4005	0.001279	0.01344	0.3385	1.485	10001	500000
delta[3]	0.8565	0.5797	0.002075	0.04612	0.7695	2.199	10001	500000
delta[4]	4.557	0.7806	0.002791	2.88	4.605	5.96	10001	500000
delta[5]	0.9683	0.6314	0.001778	0.05662	0.8845	2.405	10001	500000
delta[6]	1.048	0.6514	0.001226	0.06984	0.973	2.504	10001	500000
mu[1]	-0.1484	0.4775	0.001551	-1.181	-0.1174	0.7098	10001	500000
mu[2]	0.2987	0.4397	0.00115	-0.5765	0.2995	1.172	10001	500000
mu[3]	1.155	0.5353	0.001638	0.2069	1.119	2.314	10001	500000
mu[4]	5.712	0.5521	0.001483	4.542	5.742	6.714	10001	500000
mu[5]	6.681	0.4983	8.5E-4	5.701	6.679	7.666	10001	500000
mu[6]	7.729	0.5604	6.954E-4	6.707	7.701	8.916	10001	500000
sigma	1.062	0.2289	5.746E-4	0.7249	1.025	1.61	10001	500000

Example 9 (cont.)