IE 361 Module 4

Metrology Applications of Some Intermediate Statistical Methods for Separating Components of Variation

Reading: Section 2.2 *Statistical Quality Assurance for Engineers* (Section 2.3 of Revised *SQAME*)

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A Relatively Simple First Method for Separating Process and Measurement Variation

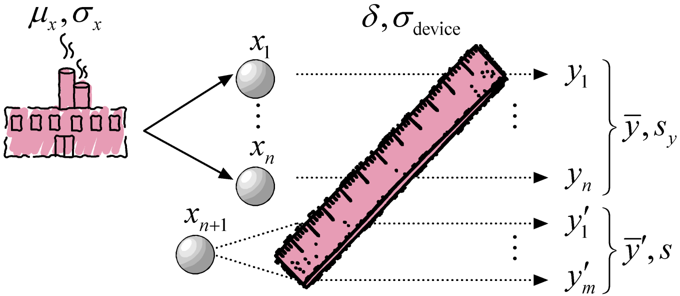
In Module 2 we observed that

1. repeated measurement of a single measurand with a single device allows one to estimate σ_{device}, and
2. single measurements made on multiple measurands from a stable process using a linear device allow one to estimate
$\sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}$

and remarked that these facts might allow one to somehow find a way to estimate σ_x (a process standard deviation) alone. Our first goal in this module is to provide one simple method of doing this.

The next figure illustrates a data collection plan that combines the elements 1. and 2. above.

One Method of Separating Process and Measurement Variation



$$y_i\text{'s} \sim \text{ind}\left(\mu_x + \delta, \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}\right) \quad \text{independent of}$$

$$y'_i\text{'s} \sim \text{ind}\left(x_{n+1} + \delta, \sigma_{\text{device}}\right)$$

Figure: One Possible Data Collection Plan for Estimating a Process Standard Deviation, σ_x

One Method of Separating Process and Measurement Variation

Here we will use the notation y for the single measurements on n items from the process and the notation y' for the m repeat measurements on a single measurand. The sample standard deviation of the y's, s_y, is a natural empirical approximation for $\sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}$ and the sample standard deviation of the y''s, s, is a natural empirical approximation for σ_{device}. That suggests that one estimate the process standard deviation with

$$\widehat{\sigma_x} = \sqrt{\max\left(0, s_y^2 - s^2\right)} \tag{1}$$

as indicated in display (2.3), page 20 of *SQAME*. (The maximum of 0 and $s_y^2 - s^2$ under the root is there simply to ensure that one is not trying to take the square root of a negative number in the rare case that s exceeds s_y.)

One Method of Separating Process and Measurement Variation

$\widehat{\sigma_x}$ is not only a sensible single number estimate of σ_x, but can also be used to make approximate confidence limits for the process standard deviation. The so-called Satterthwaite approximation suggests that one use

$$\widehat{\sigma_x}\sqrt{\frac{\hat{\nu}}{\chi^2_{\text{upper}}}} \text{ and } \widehat{\sigma_x}\sqrt{\frac{\hat{\nu}}{\chi^2_{\text{lower}}}} \text{ as limits for } \sigma_x$$

where appropriate "approximate degrees of freedom" are

$$\hat{\nu} = \frac{\widehat{\sigma_x}^4}{\frac{s_y^4}{n-1} + \frac{s^4}{m-1}}$$

One Method of Separating Process and Measurement Variation

Example 4-1

In Module 2, we considered $m = 5$ measurements made by a single analyst on a single physical sample of material using a particular assay machine that produced $s = .0120$. Suppose that subsequently, samples from $n = 20$ different batches are analyzed and $s_y = .0300$. An estimate of real process standard deviation (uninflated by measurement variation) is then

$$\widehat{\sigma_x} = \sqrt{\max\left(0, s_y^2 - s^2\right)} = \sqrt{\max\left(0, (.0300)^2 - (.0120)^2\right)} = .0275$$

and this value can used to make confidence limits. The Satterthwaite "approximate degrees of freedom" are

$$\hat{\nu} = \frac{\widehat{\sigma_x}^4}{\frac{s_y^4}{n-1} + \frac{s^4}{m-1}} = \frac{(.0275)^4}{\frac{(.0300)^4}{19} + \frac{(.0120)^4}{4}} = 11.96$$

and rounding down to $\hat{\nu} = 11$, an approximate 95% confidence interval for the real process standard deviation, σ_x, is

$$\left(.0275\sqrt{\frac{11}{21.920}}, .0275\sqrt{\frac{11}{3.816}} \right) \quad \text{i.e.} \quad (.0195, .0467)$$

One Way Random Effects Models and Associated Inference

One of the basic models of intermediate statistical methods is the so-called "one-way random effects model" for I samples of observations

$$\begin{array}{c} y_{11}, y_{12}, \ldots, y_{1n_1} \\ y_{21}, y_{22}, \ldots, y_{2n_2} \\ \vdots \\ y_{I1}, y_{I2}, \ldots, y_{In_I} \end{array}$$

This model says that the observations may be thought of as

$$y_{ij} = \mu_i + \epsilon_{ij}$$

where the ϵ_{ij} are independent normal random variables with mean 0 and standard deviation σ, while the I values μ_i are independent normal random variables with mean μ and standard deviation σ_μ (independent of the ϵ's). (One can think of I means μ_i drawn at random from a normal distribution of μ_i's, and subsequently observations y generated from I different normal populations with those means and a common standard deviation.)

One Way Random Effects Models and Associated Inference

In this model, the 3 parameters are σ (the "within group" standard deviation), σ_μ (the "between group" standard deviation), and μ (the overall mean). The squares of the standard deviations are called "variance components" since for any particular observation, the laws of expectation and variance imply that

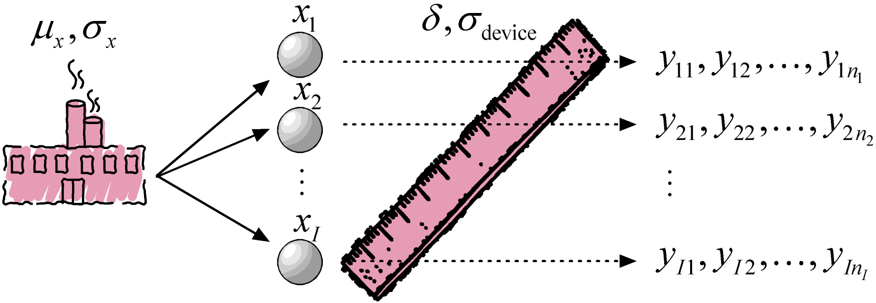
$$\mu_y = \mu + 0 = \mu \text{ and } \sigma_y^2 = \sigma_\mu^2 + \sigma^2$$

(i.e. σ_μ^2 and σ^2 are components of the variance of y).

Two quality assurance/metrological contexts where this model can be helpful are where

- multiple measurands from a stable process are each measured multiple times on the same device
- a single measurand is measured multiple times on multiple devices

These two scenarios and the accompanying parameter values are illustrated in the next two figures.



$$"\mu" = \mu_x + \delta, "\sigma_\mu" = \sigma_x, \text{ and } "\sigma" = \sigma_{\text{device}}$$

Figure: Cartoon Illustrating Multiple Measurands from a Stable Process Each Measured Multiple Times With the Same (Linear) Device

One Way Random Effects Models and Associated Inference

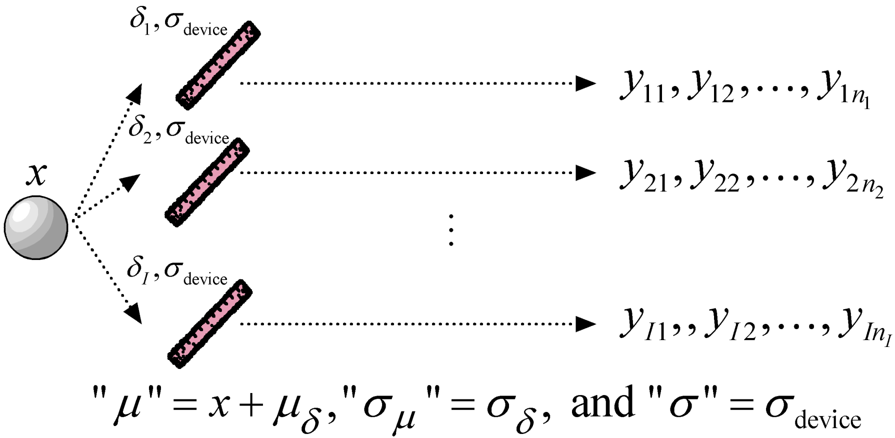


Figure: Cartoon Illustrating a Single Measurand Measured Multiple Times With Multiple Devices

There are well established (but not altogether simple) methods of inference associated with the one-way random effects model, that can be applied to make confidence intervals for the model parameters (and inferences of practical interest in metrological applications). Some of these are based on so-called ANOVA methods and the one-way ANOVA identity that says

$$\sum_{i,j} \left(y_{ij} - \overline{y}_{..}\right)^2 = \sum_{i} n_i \left(\overline{y}_{i.} - \overline{y}_{..}\right)^2 + \sum_{i,j} \left(y_{ij} - \overline{y}_{i.}\right)^2$$

or

$$SSTot = SSTr + SSE$$

For example, with $n = \sum n_i$, the quantity

$$\hat{\sigma} = \sqrt{MSE} = \sqrt{\frac{SSE}{n - I}}$$

is a square root of a (weighted) average of the I sample variances and can be used to make confidence limits for σ as

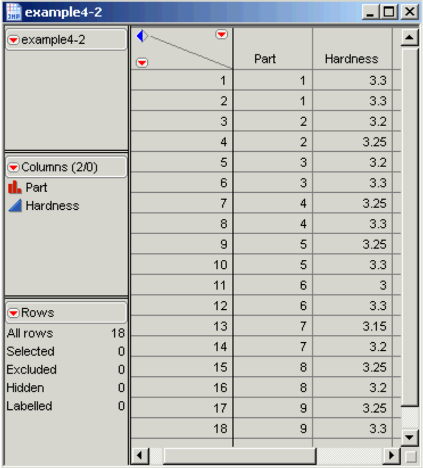
$$\hat{\sigma}\sqrt{\frac{n-I}{\chi^2_{\text{upper}}}} \quad \text{and} \quad \hat{\sigma}\sqrt{\frac{n-I}{\chi^2_{\text{lower}}}}$$

where the appropriate degrees of freedom are $\nu = n - I$. And, although we won't illustrate them here, the Satterthwaite approximation can be used to make approximate confidence limits for σ_μ.

Operationally, the most efficient way to make inferences based on the one way random effects model is to use a high quality statistical package like JMP and rely on its implementation of the best known methods of estimation of the parameters σ, σ_μ, and μ. We proceed to illustrate that possibility in a metrological application.

Consider the case of Problem 2.10, pages 50-51 of *SQAME*, and in particular the two hardness measurements made on each of the $I = 9$ parts by Operator A. This is a scenario of the type illustrated on panel 10. The following series of figures shows first a JMP data sheet for this example (note that part is a nominal variable and hardness is a continuous variable), then the dialogue box for a Fit Model procedure appropriate here (the part effect has been made a random effect by using the menu under the red triangle by "attributes" in the dialogue box), and finally a JMP report for the analysis, showing confidence limits for σ_x^2 ($= \sigma_\mu^2$ here) and for σ_{device}^2 ($= \sigma^2$ here).

What is clear from this analysis is that this is a case where part-to-part variation in hardness (measured by σ_x) is small enough and poorly determined enough in comparison to basic measurement noise (measured by σ_{device}) that it is impossible to really tell its size.



	Part	Hardness
1	1	3.3
2	1	3.3
3	2	3.2
4	2	3.25
5	3	3.2
6	3	3.3
7	4	3.25
8	4	3.3
9	5	3.25
10	5	3.3
11	6	3
12	6	3.3
13	7	3.15
14	7	3.2
15	8	3.25
16	8	3.2
17	9	3.25
18	9	3.3

Figure: JMP Data Sheet for Example 4-2 (Data From Page 51 of *SQAME*)

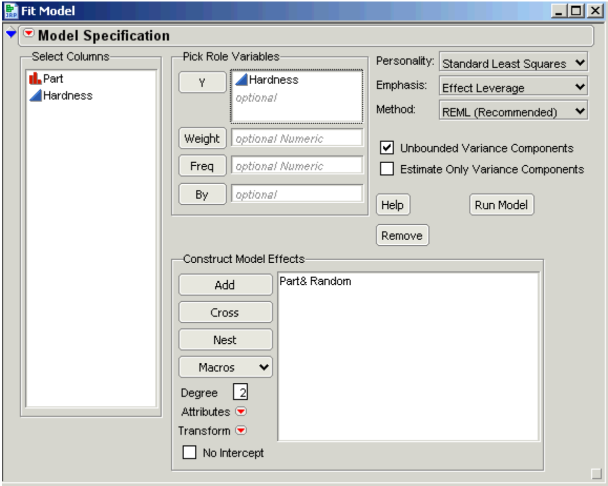
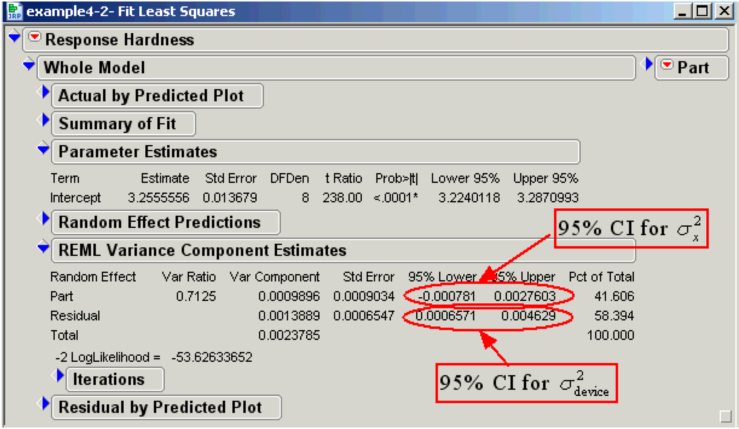


Figure: `JMP Fit Model` Dialogue Box for Example 4-2



Term	Estimate	Std Error	DFDen	t Ratio	Prob>|t|	Lower 95%	Upper 95%
Intercept	3.2555556	0.013679	8	238.00	<.0001*	3.2240118	3.2870993

Random Effect	Var Ratio	Var Component	Std Error	95% Lower	95% Upper	Pct of Total
Part	0.7125	0.0009896	0.0009034	-0.000781	0.0027603	41.606
Residual		0.0013889	0.0006547	0.0006571	0.004629	58.394
Total		0.0023785				100.000

Figure: JMP Report for Example 4-2

Consider the case of Problem 2.12, page 52 of *SQAME*, and in particular the three weight measurements made on piece 1 by each of the $I = 5$ operators. This is a scenario of the type illustrated in panel 11 and further illustrates the concepts of "repeatability" (device) variation and "reproducibility" (operator-to-operator) variation first discussed in Module 3.

The following series of figures shows first a JMP data sheet for this example (note that operator is a nominal variable and weight is a continuous variable), then the dialogue box for a Fit Model procedure appropriate here (the operator effect has been made a random effect by using the menu under the red triangle by "attributes" in the dialogue box), and finally a JMP report for the analysis, showing confidence limits for σ_δ^2 ($= \sigma_\mu^2$ here) and for σ_{device}^2 ($= \sigma^2$ here) example4-3

example4-3

Columns (2/0)
Operator
Weight

Rows
All rows 15
Selected 0
Excluded 0
Hidden 0
Labelled 0

	Operator	Weight
1	1	3.481
2	1	3.477
3	1	3.47
4	2	3.448
5	2	3.472
6	2	3.47
7	3	3.485
8	3	3.464
9	3	3.477
10	4	3.475
11	4	3.472
12	4	3.473
13	5	3.472
14	5	3.47
15	5	3.474

Figure: JMP Data Sheet for Example 4-3 (Data From Page 52 of *SQAME*)

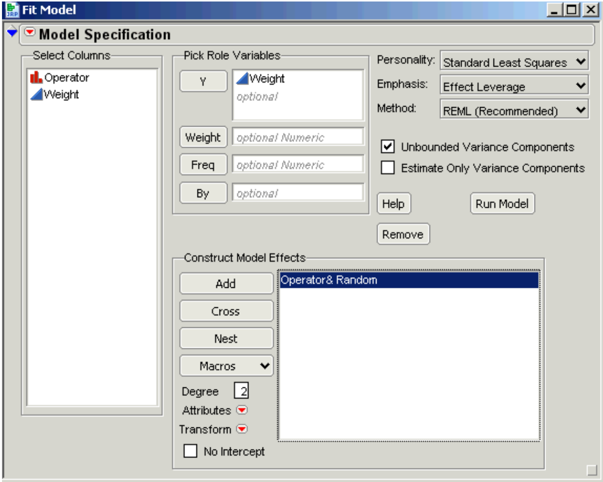
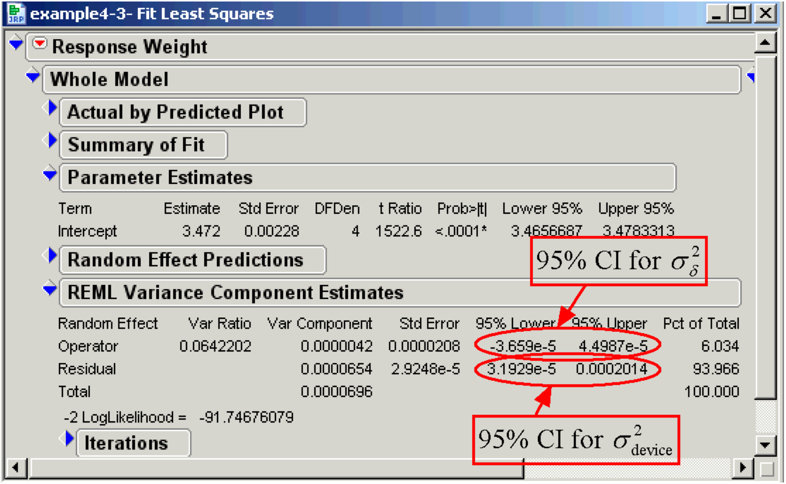


Figure: `JMP Fit Model` Dialogue Box for Example 4-3



example4-3- Fit Least Squares

Response Weight

Whole Model

Actual by Predicted Plot

Summary of Fit

Parameter Estimates

Term	Estimate	Std Error	DFDen	t Ratio	Prob>|t|	Lower 95%	Upper 95%
Intercept	3.472	0.00228	4	1522.6	<.0001*	3.4656687	3.4783313

Random Effect Predictions

REML Variance Component Estimates

Random Effect	Var Ratio	Var Component	Std Error	95% Lower	95% Upper	Pct of Total
Operator	0.0642202	0.0000042	0.0000208	-3.659e-5	4.4987e-5	6.034
Residual		0.0000654	2.9248e-5	3.1929e-5	0.0002014	93.966
Total		0.0000696				100.000

-2 LogLikelihood = -91.74676079

Iterations

Figure: JMP Report for Example 4-3

Recognizing that although the JMP report lists a negative lower confidence bound for σ_δ^2, this quantity can never be smaller than 0, we estimate with 95% confidence that

$$0 < \sigma_\delta < \sqrt{4.5 \times 10^{-5}} = .0067$$

and that

$$.0057 = \sqrt{3.2 \times 10^{-5}} < \sigma_{\text{device}} < \sqrt{.0002014} = .0142$$

and this is a case where repeatability (device) variation is clearly larger than reproducibility (operator-to-operator) variation in weight measuring. If one doesn't like the overall size of measurement variation seen in the data of panel 19, it appears that some fundamental change in equipment or how it is used will be required. Simple training of the operators aimed at making how they use the equipment more uniform (and reduction of differences between their biases) has far less potential to improve measurement precision.