

IE 361 Module 19

"Statistical" (Probabilistic) Tolerancing

Prof.s Stephen B. Vardeman and Max D. Morris

Reading: Section 5.4 *Statistical Quality Assurance Methods for Engineers*

In this module we simple probability-based methods of combining measures of input or component variability to predict output/system/overall variability. The basic tools available in this effort are

- exact formulas for means and variances of linear functions of several random variables (from Stat 231)
- approximations for means and variances of general functions of several random variables (based on linear approximations to the functions and the above)
- simple probabilistic simulations, easily done in any decent statistical package (or, heaven forbid, EXCEL)

Piecing Together Information on Variation of Components to Predict Overall Variation

Sometimes geometry or physical theory gives one an equation for a variable of interest in terms of more basic variables X, Y, \dots, Z

$$U = g(X, Y, \dots, Z)$$

and an issue of interest is how one might infer the level of variation to be seen in U from given information on the level of variation in the inputs.

Example 19-1 Example 5.9 of *SQAME* is a nice "tolerance stack-up" example. A company wished to pack 4 units of product in a carton and was experiencing difficulty in packaging. The linear size of the carton interior (Y) and exterior

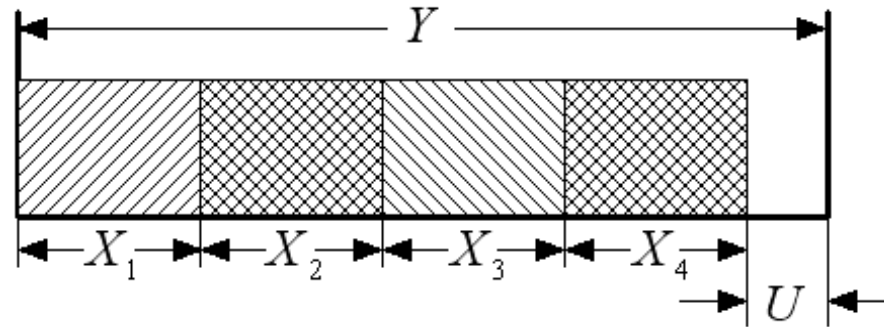


Figure 1: Cartoon of a Packaging Problem (Example 5.9 of *SQAME*)

linear sizes of four packages (X 's), together with the carton "head space" (U) are illustrated in Figure 1.

In this problem, simple geometry reveals that

$$U = Y - X_1 - X_2 - X_3 - X_4$$

Example 19-2 Example 5.8 of *SQAME* is a simple circuit example involving an assembly of 3 resistors. Figure 2 illustrates this situation.

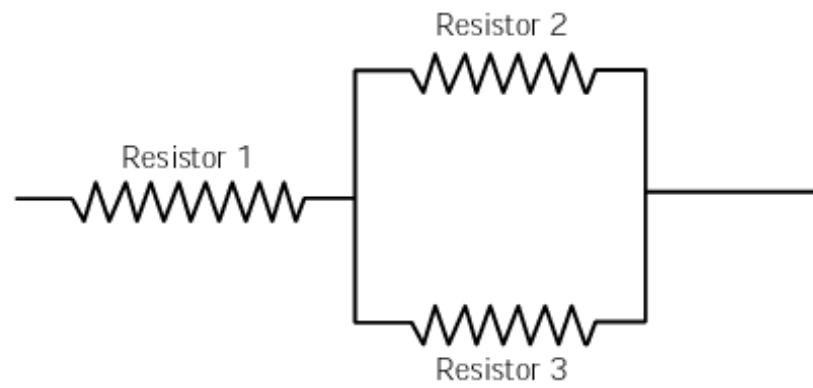


Figure 2: A Schematic for Example 5.8 of *SQAME*

In this problem, elementary physics produce an equation for assembly resistance, R

$$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$$

Example 19-3 Figure 3 concerns a "close-to-realistic" tolerancing problem faced by an engineer who must set tolerances on various geometric features of a car door assembly with the end goal of creating a uniform (around the door) gap between the door and the body of an automobile on which it is to be hung.

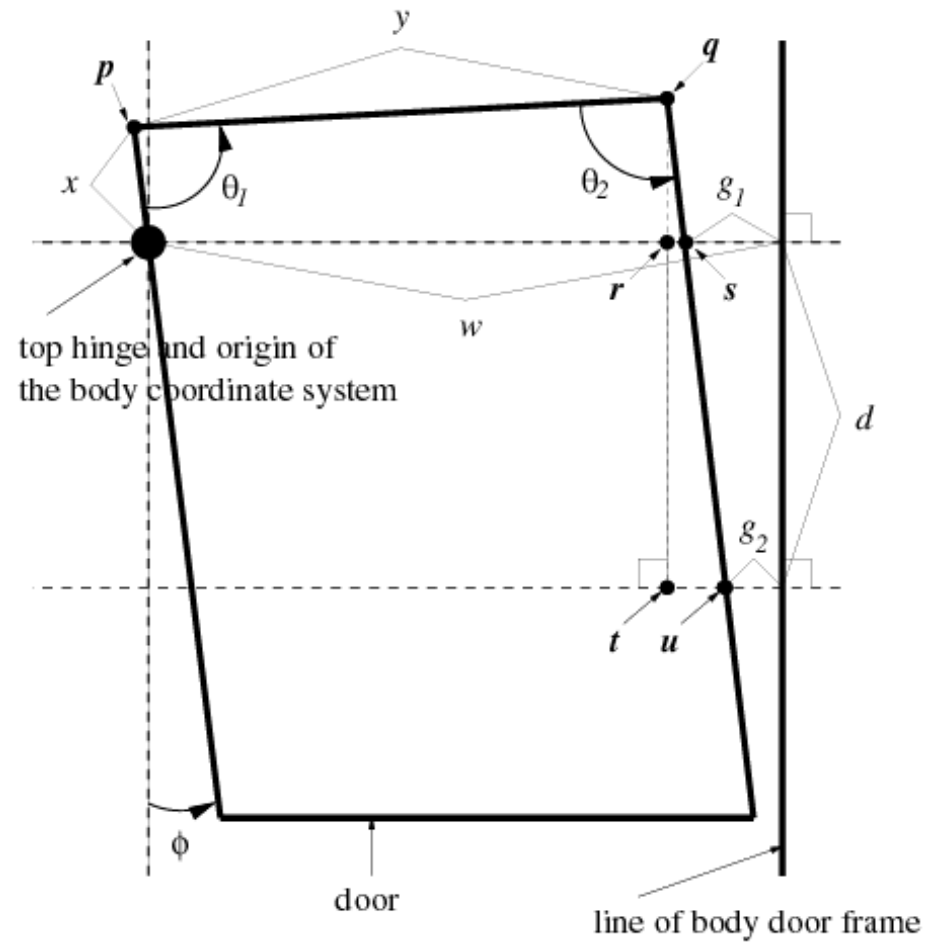


Figure 3: Cartoon of a Door Tolerancing Problem

Some careful plane geometry and trigonometry applied to this situation produce the following set of equations that in the end express the gaps g_1 and g_2 at the elevation of the top hinge and a distance d below that hinge in terms of $x, w, \theta_1, y, \theta_2$, and ϕ (which are all quantities on which a design engineer would need to set tolerances)

$$\mathbf{p} = (-x \sin \phi, x \cos \phi)$$

$$\mathbf{q} = \mathbf{p} + \left(y \cos \left(\phi + \left(\theta_1 - \frac{\pi}{2} \right) \right), y \sin \left(\phi + \left(\theta_1 - \frac{\pi}{2} \right) \right) \right)$$

$$\mathbf{s} = (q_1 + q_2 \tan(\phi + \theta_1 + \theta_2 - \pi), 0)$$

$$\mathbf{u} = (q_1 + (q_2 + d) \tan(\phi + \theta_1 + \theta_2 - \pi), -d)$$

$$g_1 = w - s_1$$

$$g_2 = w - u_1$$

That is, though we have not written them out explicitly here, there are 2 functions of the inputs $x, w, \theta_1, y, \theta_2$, and ϕ that produce the 2 gap values.

In these three examples, then, the question is how one should expect variation in the inputs to be reflected in the outputs. If one is willing to model inputs X, Y, \dots, Z as independent random variables

- equations (5.23) and (5.24) of *SQAME* are straight from Stat 231 and say that for g linear, i.e. where for constants a_0, a_1, \dots, a_k ,

$$U = a_0 + a_1X + a_2Y + \dots + a_kZ.$$

U has mean

$$\mu_U = a_0 + a_1\mu_X + a_2\mu_Y + \dots + a_k\mu_Z$$

and variance

$$\sigma_U^2 = a_1^2\sigma_X^2 + a_2^2\sigma_Y^2 + \dots + a_k^2\sigma_Z^2$$

- equations (5.26) and (5.27) of *SQAME* are based on a (Taylor theorem) "linearization" of a general g and the above relationship and say that roughly

$$\mu_U \approx g(\mu_X, \mu_Y, \dots, \mu_Z)$$

and

$$\sigma_U^2 \approx \left(\frac{\partial g}{\partial x}\right)^2 \sigma_X^2 + \left(\frac{\partial g}{\partial y}\right)^2 \sigma_Y^2 + \dots + \left(\frac{\partial g}{\partial z}\right)^2 \sigma_Z^2$$

where the partial derivatives are evaluated at the point $(\mu_X, \mu_Y, \dots, \mu_Z)$

- simple probabilistic simulations can be used to approximate the distribution of U based on some choice of distributions for the inputs in very straightforward fashion

Before proceeding to illustrate these 3 methods, there are several points to be made. In the first place, notice that since in the case of a linear g , the a 's are exactly the partial derivatives of U with respect to the input variables, the general approximation produces the exact result in case g is exactly linear. Second, note that while we will see that simulation is completely straightforward and indeed almost mindless to carry out, there will be occasions where it is desirable to use the formulas. In particular, it's possible to look at a term like

$$\left(\frac{\partial g}{\partial x}\right)^2 \sigma_X^2$$

in an approximation for σ_U^2 as the part of the variation in U traceable to variation in the input X . Finally, we observe that the approximation for σ_U^2 is "qualitatively right." The variability in U must depend upon both 1) how variable the inputs are and 2) what the rates of change of output with respect to inputs are. (These two are measured respectively by the variances and the derivatives.) Figure 4 illustrates the importance of the derivatives in determining how variance is transmitted through g .

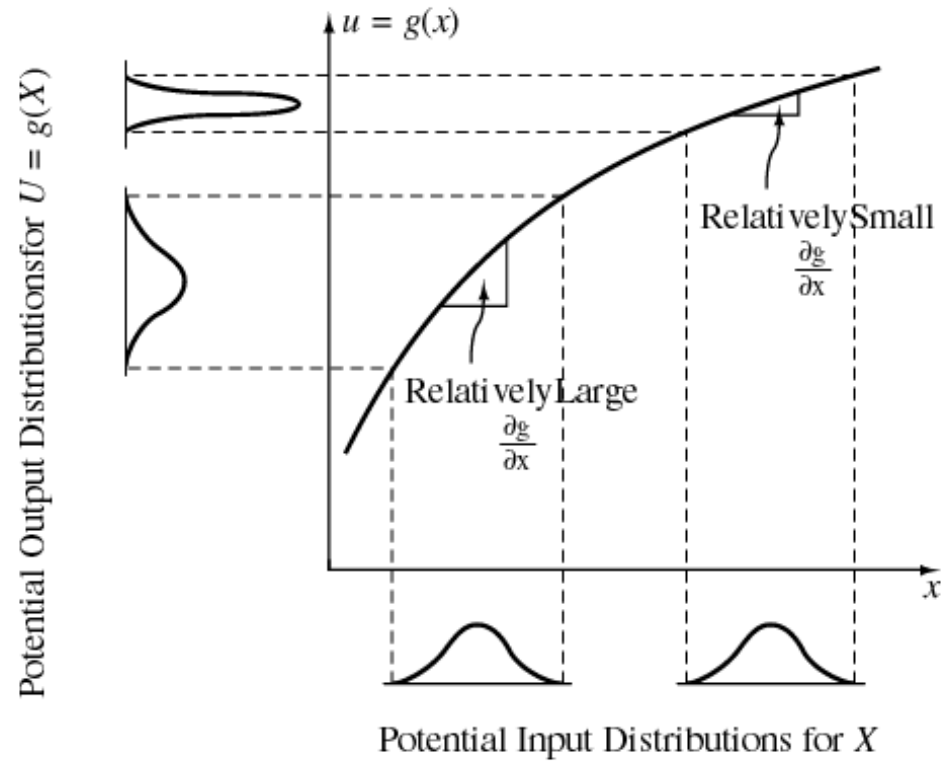


Figure 4: Cartoon Illustrating the Importance of Rates of Change in Determining Variance Transmission (g a Function of One Input)

Example 19-1 continued Returning to Example 5.9, note that one may apply the exact form for the variance of U for a linear g to obtain

$$\sigma_U^2 = 1^2\sigma_Y^2 + (-1)^2\sigma_{X_1}^2 + (-1)^2\sigma_{X_2}^2 + (-1)^2\sigma_{X_3}^2 + (-1)^2\sigma_{X_4}^2$$

The IE 361 team estimated (via some data collection) that $\mu_Y \approx 9.556$ in and $\sigma_Y \approx .053$ in and $\mu_X \approx 2.577$ in and $\sigma_X = .061$ in. This leads then to

$$\sigma_U^2 \approx (.053)^2 + 4(.061)^2 = .0177 \text{ in}^2$$

and

$$\sigma_U = \sqrt{.0177} = \sigma_U = .133 \text{ in}$$

This together with the fact that

$$\mu_U \approx 9.556 - 4(2.577) = -.7520$$

(the mean head space was about *negative* 3/4 in) conspired to create the packing problems. Some adjustment (downward) in μ_X and/or (upward) in μ_Y

(potentially together with reductions in σ_X and σ_Y) were going to be necessary to eliminate packing problems. Notice that the kind of analysis here would allow the company to see the impact of various changes (that come with corresponding costs).

Example 19-2 continued Returning to Example 5.8 of *SQAME* note that with

$$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$$

it is the case that

$$\frac{\partial R}{\partial R_1} = 1, \frac{\partial R}{\partial R_2} = \frac{R_3^2}{(R_2 + R_3)^2}, \text{ and } \frac{\partial R}{\partial R_3} = \frac{R_2^2}{(R_2 + R_3)^2}$$

So, for example, if one is making resistor assemblies using R_1 with mean 100Ω and standard deviation 2Ω , and both R_2 and R_3 with mean 200Ω and standard

deviation 4Ω , the approximations for the mean and variance of R provide

$$\mu_R \approx 100 + \frac{200 \cdot 200}{(200 + 200)^2} = 200 \Omega$$

and

$$\sigma_R \approx \sqrt{1^2 (2)^2 + \left(\frac{1}{4}\right)^2 (4)^2 + \left(\frac{1}{4}\right)^2 (4)^2} = 2.45 \Omega$$

(Of course, engineering mathematics software like MathCad can be used in problems like this, but this particular calculus is easy enough to do without resort to that kind of a tool.)

To illustrate how easily simulations can be used to provide answers in problems of the type under discussion in this module, the next figure shows what happens if one uses normal distributions with means and standard deviations as above to simulate 1000 values of R and computes summary statistics from those.

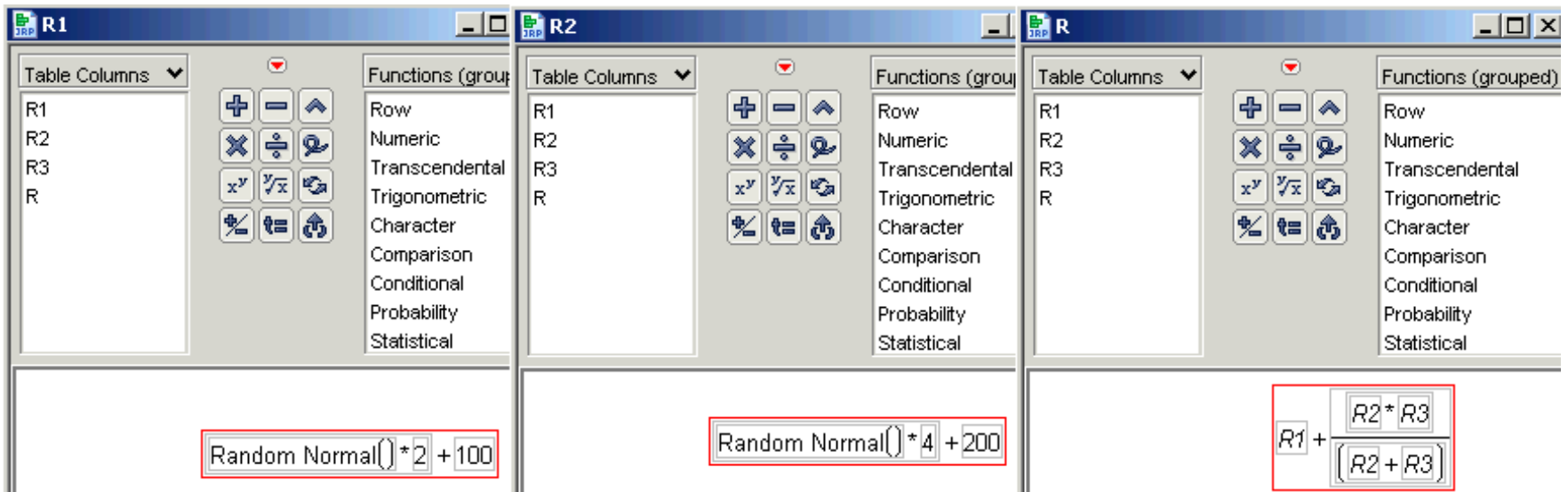


Figure 5: JMP Dialogue Boxes for Setting Up Columns for Simulation of R

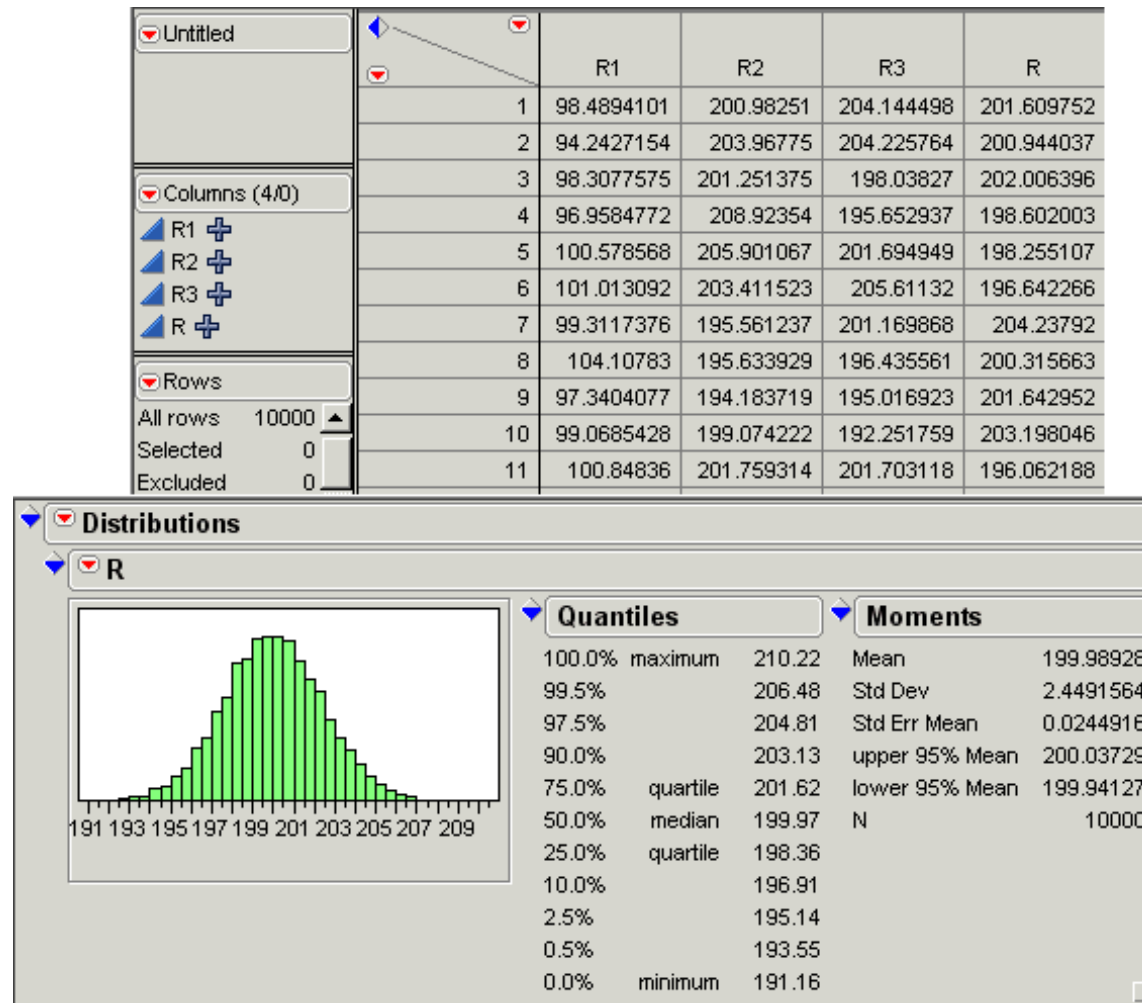


Figure 6: JMP Data Sheet and Report for Simulation of R

Notice that the report based on 10,000 simulated values of R (from normal inputs) is in substantial agreement with the earlier analysis giving approximate values for μ_R and σ_R .

Example 19-3 continued To illustrate how simply something as apparently complicated as the car door tolerancing problem can be handled using simulation, the following is a bit of Minitab code and corresponding output for one set of means and standard deviations for $x = c1$, $y = c2$, $w = c3$, $\phi = c4$, $\theta_1 = c5$, $\theta_2 = c6$, at $d = 40$ (in units of cm and rad).

```
MTB > Random 1000 c1;
```

```
SUBC> Normal 20 .01.
```

```
MTB > Random 1000 c2;
```

```
SUBC> Normal 90. .01.
```

```
MTB > Random 1000 c3;
```

```
SUBC> Normal 90.4 .01.
```

```
MTB > Random 1000 c4;
```

```
SUBC> Normal 0 .001.
```

```
MTB > Random 1000 c5 c6;
```

```
SUBC> Normal 1.570796 .001.
```

```
MTB > let c7=-c1*sin(c4)
```

MTB > let c8=c1*cos(c4)

MTB > let c9=c7+c2*cos(c4+(c5-1.570796))

MTB > let c10=c8+c2*sin(c4+(c5-1.570796))

MTB > let c11=c9+c10*tan(c4+c5+c6-3.141593)

MTB > let c12=c9+(c10+40)*tan(c4+c5+c6-3.141593)

MTB > let c13=c3-c11

MTB > let c14=c3-c12

MTB > Describe 'g1' 'g2'.

Descriptive Statistics

Variable	N	Mean	Median	StDev
g1	1000	0.40053	0.40035	0.03078
g2	1000	0.40138	0.39775	0.09153

Variable	Minimum	Maximum	Q1	Q3
g1	0.31109	0.51216	0.37877	0.42168
g2	0.12174	0.75907	0.34043	0.46280

Where they are applicable, these methods of "statistical tolerancing" are powerful tools for understanding the consequences of various levels of input variation on the output variation of a system.