

# IE 361 Module 15

## The Average Run Length Concept

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Reading: Section 3.5, *Statistical Quality Assurance Methods for Engineers*

The general question addressed here is "How does one quantify the likely performance of a process monitoring scheme?"

## **Quantifying What a Monitoring Scheme Can Be Expected to Do: the ARL Concept**

Once one begins to contemplate alternative schemes for issuing out-of-control signals based on process-monitoring data, the need quickly arises to *quantify* what a given scheme might be expected to do. For example, what are the pros and cons of adding the Western Electric set of alarm rules to a control charting

scheme? The most effective means known for making this kind of prediction is the "Average Run Length" (ARL) notion.

It is useful to adopt the notation

$T$  = the period at which a process-monitoring scheme first signals

$T$  is called the (random) **run length** for the scheme. The probability distribution of  $T$  is called the **run length distribution**, and the mean or average value of this distribution is called the **Average Run Length** (ARL) for the process-monitoring scheme. That is,

$$ARL = \mu_T$$

When one is setting up a process monitoring scheme, it is desirable that it produce a large ARL when the process is stable at standard values for process parameters and small ARLs under other conditions.

Evaluating ARLs is not usually elementary. But there is one circumstance where an explicit formula for ARLs is possible and we can illustrate the meaning and usefulness of the ARL concept in elementary terms. That is the situation where

- the process-monitoring scheme employs only the single alarm rule "signal the first time that a point  $Q$  plots outside control limits," and
- it is sensible to think of the process as physically stable (though perhaps not at standard values for process parameters).

Under the second condition, the values  $Q_1, Q_2, Q_3, \dots$  can be modeled as random draws from a fixed distribution, and the notation

$$q = P[Q_1 \text{ plots outside control limits}]$$

will prove useful. In this simplest of cases, it follows that

$$ARL = \frac{1}{q}$$

**Example 15-1** (Some ARLs for Shewhart  $\bar{x}$  charts) Consider finding ARLs for a standards given Shewhart  $\bar{x}$  chart based on samples of size  $n = 5$ . Note that if standard values for the process mean and standard deviation are respectively  $\mu$  and  $\sigma$ , the relevant control limits are

$$UCL_{\bar{x}} = \mu + 3\frac{\sigma}{\sqrt{5}} \quad \text{and} \quad LCL_{\bar{x}} = \mu - 3\frac{\sigma}{\sqrt{5}}$$

Thus

$$q = P \left[ \bar{x} < \mu - 3\frac{\sigma}{\sqrt{5}} \quad \text{or} \quad \bar{x} > \mu + 3\frac{\sigma}{\sqrt{5}} \right]$$

First suppose that "all is well" and the process is stable at standard values of the process parameters. Then  $\mu_{\bar{x}} = \mu$  and  $\sigma_{\bar{x}} = \sigma/\sqrt{5}$  and if the process output is normal, so also is the random variable  $\bar{x}$ . Thus

$$q = 1 - P \left[ \mu - 3\frac{\sigma}{\sqrt{5}} < \bar{x} < \mu + 3\frac{\sigma}{\sqrt{5}} \right] = 1 - P \left[ -3 < \frac{\bar{x} - \mu}{\sigma/\sqrt{5}} < 3 \right]$$

can be evaluated using the fact that

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{5}}$$

is standard normal. Using a normal table (with an additional significant digit beyond what is typical) it is possible to establish that

$$q = 1 - P[-3 < Z < 3] = .0027$$

to 4 digits. Therefore, it follows that

$$ARL = \frac{1}{.0027} = 370$$

The interpretation is that when all is OK (i.e., the process is stable and parameters are at their standard values), the  $\bar{x}$  chart will issue (false alarm) signals on average only once every 370 plotted points.

In contrast, consider the possibility that (while the process standard deviation is at its standard value) the process mean is one standard deviation above its standard value. In these circumstances one still has  $\sigma_{\bar{x}} = \sigma/\sqrt{5}$ , but now  $\mu_{\bar{x}} = \mu + \sigma$  ( $\mu$  and  $\sigma$  are still the standard values of respectively the process mean and standard deviation). Then,

$$\begin{aligned}
 q &= 1 - P \left[ \mu - 3\frac{\sigma}{\sqrt{5}} < \bar{x} < \mu + 3\frac{\sigma}{\sqrt{5}} \right] \\
 &= 1 - P \left[ \frac{\mu - 3\frac{\sigma}{\sqrt{5}} - (\mu + \sigma)}{\sigma/\sqrt{5}} < \frac{\bar{x} - (\mu + \sigma)}{\sigma/\sqrt{5}} < \frac{\mu + 3\frac{\sigma}{\sqrt{5}} - (\mu + \sigma)}{\sigma/\sqrt{5}} \right] \\
 &= 1 - P[-5.24 < Z < .76] \\
 &= .2236
 \end{aligned}$$

The following figure illustrates the calculation being done here and shows the roughly 22% chance that under these circumstances the sample mean will plot outside  $\bar{x}$  chart control limits.

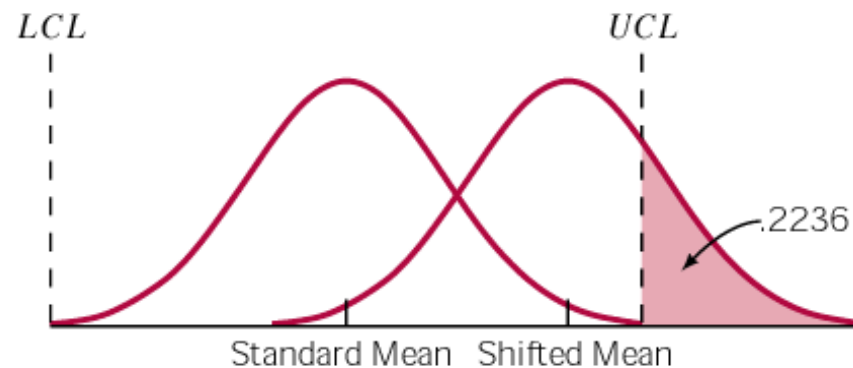


Figure 1:  $q$  in a (Particular) Scenario Where  $\mu_{\bar{x}} = \mu + \sigma$  (Distribution of  $\bar{x}$  Pictured)

So for this case where the process mean is shifted from its standard value by 1 process standard deviation,

$$ARL = \frac{1}{.2236} = 4.5$$

That is, if the process mean is off target by as much as one process standard deviation, then it will take on average only 4.5 samples of size  $n = 5$  to detect this kind of misadjustment.

This example should agree completely with intuition about "how things should be." It says that when a process is on target, one can expect long periods between signals from an  $\bar{x}$  chart. On the other hand, should the process mean shift off target by a substantial amount, there will typically be quick detection of that change.

It is possible to produce elementary formulas for ARLs only for very simple cases. Where the rules used to convert observed values  $Q_1, Q_2, Q_3, \dots$  into out-of-control signals or the probability model for these variables are more complicated

than the combination of the "one point outside control limits" rule and the stable process model, elementary computations are rare. We will soon make use of the ARL idea for some monitoring schemes more complicated than Shewhart charts. But the methods we'll use are not based on explicit formulas, but rather on ARL tables prepared by researchers who have done the more advanced calculations. It is not necessary to understand the details needed to produce the tables in order to use them and to appreciate what an ARL says about a monitoring scheme.

A result that should be mentioned here concerns ARLs for control charts when some of the "special checks" for patterns discussed earlier are used. Champ and Woodall in a 1987 *Technometrics* paper gave ARLs for monitoring schemes that use various combinations of the four Western Electric alarm rules. As an example of conclusions that were reached, consider ARLs for an  $\bar{x}$  chart. We have seen that the "all OK" ARL for a scheme using only the "one point outside

$3\sigma_{\bar{x}}$  control limits" rule is about 370. When all four Western Electric rules are employed simultaneously, Champ and Woodall found that the "all OK" ARL is much less than 370 (and what naive users of the rules might expect), namely approximately 92. The reduction from 370 to 92 shows the effect (in terms of increased frequency of false alarms) of allowing other indicators of process change in addition to individual points outside control limits.

**Example 15-2** A useful contrast to the ARL computations for  $\bar{x}$  charts can be made by considering competing  $p$  charts. For example, in the IE 361 Deming drama, it is typical to say that specifications on values drawn from the bag are  $L = 3$  and  $U = 7$ . Let's ignore the basic discreteness of the bag being sampled and compare operation of an  $\bar{x}$  chart for  $n = 5$  and standards  $\mu = 5$  and  $\sigma = 1.715$  to the operation of a  $p$  chart that calls an observation  $x$  nonconforming if

$$x < 3 \text{ or } x > 7$$

Note that since

$$z_1 = \frac{3 - 5}{1.715} = -1.17 \quad \text{and} \quad z_2 = \frac{7 - 5}{1.715} = 1.17$$

and

$$P[Z < -1.17 \text{ or } Z > 1.17] = .2420$$

a  $p$  chart based on

$\hat{p}$  = the fraction of  $n = 5$  individuals nonconforming

will have standards given upper control limit

$$\begin{aligned} UCL_{\hat{p}} &= .2420 + 3\sqrt{\frac{.2420(1 - .2420)}{5}} \\ &= .8166 \end{aligned}$$

and no lower control limit since  $.2420 - 3\sqrt{\frac{.2420(1 - .2420)}{5}} < 0$ . This means

(since  $n = 5$  and  $\hat{p}$  can be only 0, .2, .4, .6, .8, or 1.0) that

$$\begin{aligned} q &= P [5 \text{ out of } n = 5 \text{ individuals are outside of the specifications}] \\ &= (.2420)^5 \\ &= .00083 \end{aligned}$$

So the all OK ARL for the  $p$  chart is

$$ARL = \frac{1}{.00083} = 1,205$$

On the other hand, if the process mean shifts by  $\sigma = 1.715$  (to, say,  $5 + 1.715 = 6.715$ ) since

$$z_1 = \frac{3 - 6.715}{1.715} = -2.17 \quad \text{and} \quad z_2 = \frac{7 - 6.715}{1.715} = .17$$

and

$$P [Z < -2.17 \text{ or } Z > .17] = .4475$$

and

$$\begin{aligned} q &= P [5 \text{ out of } n = 5 \text{ individuals are outside of the specifications}] \\ &= (.4475)^5 \\ &= .0179 \end{aligned}$$

So the ARL for the  $p$  chart is

$$ARL = \frac{1}{.0179} = 56$$

Examples 15-1 and 15-2 can be summarized in tabular form as below

	$\bar{x}$ Chart	$p$ Chart
$\mu = 5$ ARL	370	1205
$\mu = 6.715$ ARL	4.5	56

and it's clear that the  $p$  chart will not detect the shift in process mean anywhere nearly as quickly as will an  $\bar{x}$  chart based on the same sample size ( $n = 5$  in this case).