IE 361 Module 13

Control Charts for Counts ("Attributes Data")

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Reading: Section 3.3, *Statistical Quality Assurance Methods for Engineers*

In this module, we discuss the Shewhart control charts for so-called "fraction nonconforming" and "mean nonconformities per unit" contexts. These are the Shewhart p and np charts and the Shewhart u (and c) charts. These tools are easy enough to explain and use, but are typically really NOT very effective in modern applications where acceptable nonconformity rates are so small as to be stated in "parts per million."

p Charts (and np Charts)

The scenario under which a p chart or (np chart) is potentially appropriate is one where periodically groups of n items (or outcomes) from a process are

looked at and

$$X = \text{the number of outcomes among the } n \text{ that are "nonconforming"}$$

is observed. This is illustrated in the cartoon in Figure 1, where dark balls represent nonconforming outcomes.

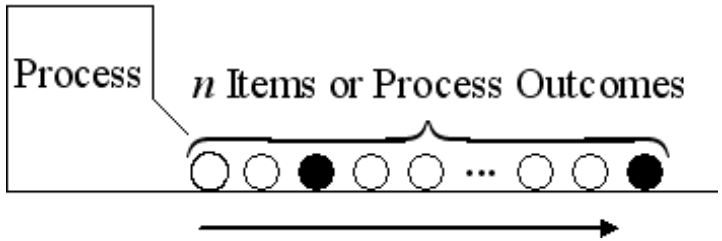

Figure 1: Cartoon Representing n Process Outcomes, X of Which are Nonconforming

In this kind of circumstance, the notation

$$\hat{p} = \frac{X}{n} = \text{the sample fraction nonconforming}$$

is standard, and

- control charts for $\hat{p}$ are called p **charts**

- control charts for X $(= n\hat{p})$ are called np **charts**

If the process producing items/outcomes is physically stable, a reasonable probability model for X (met in Stat 231) is the

$$\text{binomial } (n, p)$$

distribution, where

$$p = \text{the current probability that any particular outcome is nonconforming}$$

(the mental fiction here is that the particular n outcomes observed are a random sample of a huge pool of outcomes, a fraction p of which are nonconforming). Stat 231 facts about the binomial distribution are that

$$\mu_X = np \quad \text{and} \quad \sigma_X = \sqrt{np\,(1-p)}$$

so that (since $\hat{p} = \left(\frac{1}{n}\right) X$)

$$\mu_{\hat{p}} = p \quad \text{and} \quad \sigma_{\hat{p}} = \sqrt{\frac{p\,(1-p)}{n}}$$

These facts in turn lead to **standards given (p chart) control limits for $\hat{p}$**

$$LCL_{\hat{p}} = p - 3\sqrt{\frac{p\,(1-p)}{n}} \quad \text{and} \quad UCL_{\hat{p}} = p + 3\sqrt{\frac{p\,(1-p)}{n}}$$

and **standards given (np chart) control limits for X**

$$LCL_X = np - 3\sqrt{np\,(1-p)} \quad \text{and} \quad UCL_X = np - 3\sqrt{np\,(1-p)}$$

Example 13-1 Below are some artificially generated (using $p = .03$) $n = 400$ binomial data (except for sample 9, where a larger value of p was used) and the corresponding values of $\hat{p}$.

Sample	1	2	3	4	5	6	7	8	9	10
X	15	11	18	9	13	11	10	19	24	7
$\hat{p}$	.0375	.0275	.0450	.0225	.0325	.0275	.0250	.0475	.0600	.0175

Sample	11	12	13	14	15	16	17	18	19	20
X	9	13	17	7	10	19	11	8	8	7
$\hat{p}$	.0225	.0325	.0425	.0175	.0250	.0475	.0275	.0200	.0200	.0175

Standards given control limits for $\hat{p}$ here are

$$\begin{aligned} LCL_{\hat{p}} &= p - 3\sqrt{\frac{p\,(1-p)}{n}} \\ &= .03 - 3\sqrt{\frac{.03\,(1-.03)}{400}} = .0044 \end{aligned}$$

and

$$UCL_{\hat{p}} = .03 + 3\sqrt{\frac{.03\,(1-.03)}{400}} = .0556$$

A `JMP` plot of the corresponding control chart is in Figure 2 and shows clearly that sample 9 produces an out-of-control signal. That sample simply does not fit the "stable process with $p = .03$" model that stands behind the control limits.

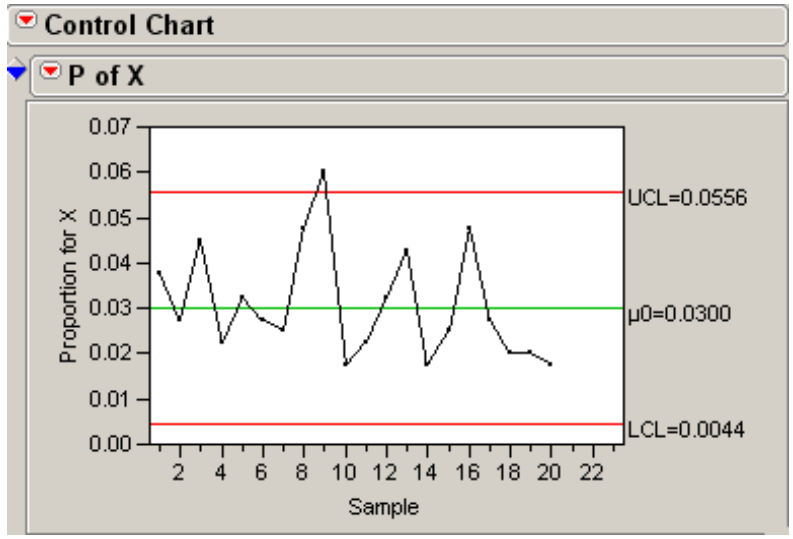

Figure 2: Standards Given ($p = .03$) p Chart for the Artificial Data

Retrospective control limits for X or $\hat{p}$ are obtained (as always) by making a provisional assumption of process stability and estimating process parameters (here, the value p). The most sensible estimate obtainable from r values $X_i = n\hat{p}_i$ based on respective sample sizes n_i is

$$\begin{aligned}\hat{p}_{\text{pooled}} &= \frac{\text{total outcomes nonconforming}}{\text{total outcomes observed}} \\ &= \frac{X_1 + X_2 + \cdots + X_r}{n_1 + n_2 + \cdots + n_r}\end{aligned}$$

(This is the arithmetic mean of the $\hat{p}_i$ in cases where the sample size is constant.)

Example 13-1 continued Returning to the artificial data, there are a total of 246 nonconforming outcomes indicated among the $20\,(400) = 8000$ outcomes represented. So

$$\hat{p}_{\text{pooled}} = \frac{246}{8000} = .0308$$

and retrospective control limits for $\hat{p}$ are

$$LCL_{\hat{p}} = .0308 - 3\sqrt{\frac{.0308\,(1 - .0308)}{400}} = .0049$$

and

$$UCL_{\hat{p}} = .0308 + 3\sqrt{\frac{.0308\,(1 - .0308)}{400}} = .0567$$

Figure 3 show that the retrospective limits do not produce a picture much different from the standards given ones used to make Figure 2. The 20 samples do not fit with a "constant p" model.

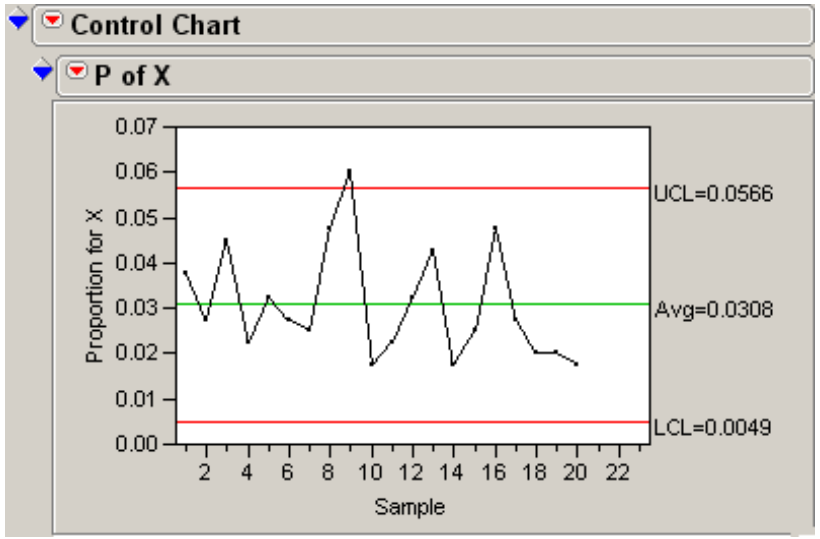

Figure 3: Retrospective p Chart for the Artificial Data

Notice that a lower control limit for a p chart CAN be positive. This sometimes seems counter-intuitive, but only to those who fail to remember that control charts are about detecting process change, NOT about product acceptability. When a point plots below a lower control limit on a p or np chart, one is alerted to unexpected/fortuitous process improvement. One wants to follow up on such lucky circumstances, looking for an assignable cause to incorporate into the process on a permanent basis.

In some cases, sample sizes vary period to period. When that happens, each new period may need a new calculation of control limits for that sample size. NOTICE that big sample sizes carry relatively larger amounts of information about current process conditions and have control limits that are tighter around a center line than small sample sizes. (With more information in a sample, less deviation from the expected value for $\hat{p}$ is required before one is fairly sure that something has changed in the process.)

Example 13-2 Below is a comparison between ($p = .03$) standards given limits for $\hat{p}$ for $n = 400$ and $n = 100$.

n	$CL_{\hat{p}}$	$LCL_{\hat{p}}$	$UCL_{\hat{p}}$
100	.03	none	.0812
400	.03	.0044	.0556

u Charts (and c Charts)

The scenario under which a u chart or (c chart) is potentially appropriate is one where periodically k inspection units of product or production from a process are looked at and

$X =$ the total number of "nonconformities" found across those k units

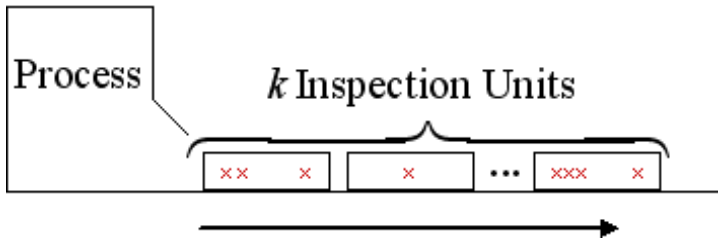

Figure 4: Cartoon of k Inspection Units of Process Output and X "Nonconformities"

is observed. This is illustrated in the cartoon in Figure 4, where "x"'s represent nonconformities.

In this kind of circumstance, the notation

$$\hat{u} = \frac{X}{k} = \text{the sample rate of nonconformities per unit}$$

is standard, and

- control charts for $\hat{u}$ are called u **charts**

- control charts for the special case where $k = 1$ and thus $\hat{u} = X$ are called c **charts**

If the process producing items/outcomes is physically stable and produces nonconformities at a rate of λ per unit, a reasonable probability model for X (met in Stat 231) is the

$$\text{Poisson } (k\lambda)$$

distribution (the Poisson distribution with mean $\mu_X = k\lambda$). A Stat 231 fact about the Poisson distribution is that its mean and variance are the same, that is that

$$\sigma_X = \sqrt{\mu_X}$$

So in the present context

$$\mu_X = k\lambda \text{ and } \sigma_X = \sqrt{k\lambda}$$

and in turn that

$$\mu_{\hat{u}} = \lambda \text{ and } \sigma_{\hat{u}} = \sqrt{\frac{\lambda}{k}}$$

These facts lead to **standards given (u chart) control limits for $\hat{u}$**

$$LCL_{\hat{u}} = \lambda - 3\sqrt{\frac{\lambda}{k}} \text{ and } UCL_{\hat{u}} = \lambda + 3\sqrt{\frac{\lambda}{k}}$$

and for the case of $k = 1$ (so that $\hat{u} = X$) **standards given (c chart) control limits for X**

$$LCL_X = \lambda - 3\sqrt{\lambda} \text{ and } LCL_X = \lambda + 3\sqrt{\lambda}$$

Example 13-3 Below are some artificially generated (using $\lambda = 2.0$) Poisson data, X. We can think of these as either counts or rates for a constant number of inspection units $k = 1$.

Sample	1	2	3	4	5	6	7	8	9	10
X	2	2	1	2	2	3	4	3	2	0

Sample	11	12	13	14	15	16	17	18	19	20
X	2	0	3	2	1	5	2	2	1	3

Standards given control limits for X would then be

$$\begin{aligned} CL_X &= 2 \\ UCL_X &= \lambda + 3\sqrt{\lambda} = 2 + 3\sqrt{2} = 6.24 \end{aligned}$$

and no lower control limit would be used since $2 - 3\sqrt{2}$ is negative. Clearly, none of the values X in the table would plot above the upper control limit and there is no indication of "lack of control"/deviation from the $\lambda = 2.0$ nonconformities per unit picture of process performance under which the above limits are developed.

To extend this example a bit, suppose that an additional 5 samples are obtained, but now the number of inspection units observed is no longer constant, and in fact that the table below is relevant.

Sample	k	X	$\hat{u} = \frac{X}{k}$	$UCL_{\hat{u}} = 2 + 3\sqrt{\frac{2}{k}}$	$LCL_{\hat{u}}$
21	1.5	2	1.33	5.67	none
22	1	1	1.00	6.24	none
23	.75	2	2.67	6.90	none
24	.5	1	2.00	8.00	none
25	3	5	1.67	5.29	none

Figure 5 is a `JMP` control chart for the whole data set, and there is no reason in these data to doubt that the nonconformity rate is constant at $\lambda = 2.0$ per inspection unit. (Note that `JMP` in ambiguous fashion indicates that $LCL = 0$. It is better practice and provides clearer intent to say that there is no lower control limit.)

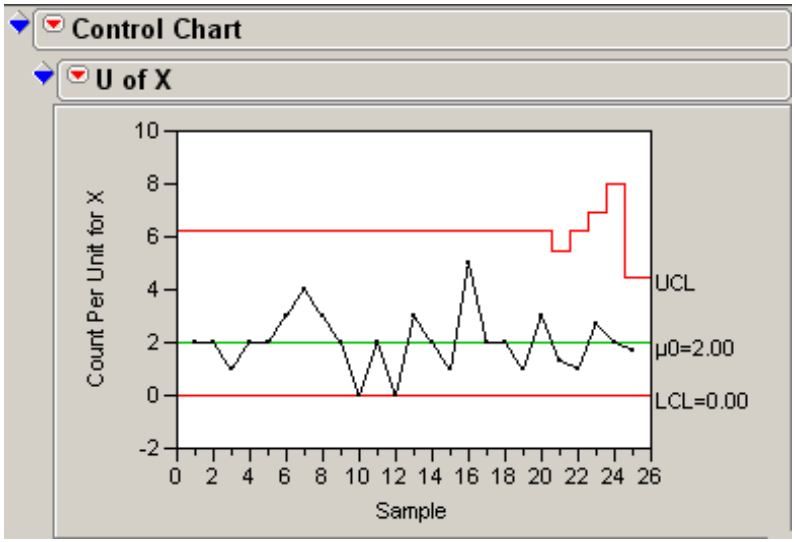

Figure 5: Standards Given ($\lambda = 2.0$) u Chart for the Artificial Data

Retrospective control limits for $\hat{u}$ are obtained (as always) by making a provisional assumption of process stability and estimating process parameters (here, the value λ). The most sensible estimate obtainable from r values X_i based on respective numbers of inspection unit k_i is

$$\begin{aligned}\hat{\lambda}_{\text{pooled}} &= \frac{\text{total nonconformities observed}}{\text{total inspection units}} \\ &= \frac{X_1 + X_2 + \cdots + X_r}{k_1 + k_2 + \cdots + k_r}\end{aligned}$$

Example 13-3 continued Returning to the artificial data, there are a total of 53 nonconforming outcomes indicated among the 26.75 inspection units. So

$$\hat{\lambda}_{\text{pooled}} = \frac{53}{26.75} = 1.98$$

and the reader can easily verify that there is negligible difference between the standards given chart in Figure 5 and the retrospective control chart that would be made using this value.

Students sometimes have a difficult time identifying whether a given application calls for a p chart or for a u chart. Figure 6 lays the two basic scenarios side by side.

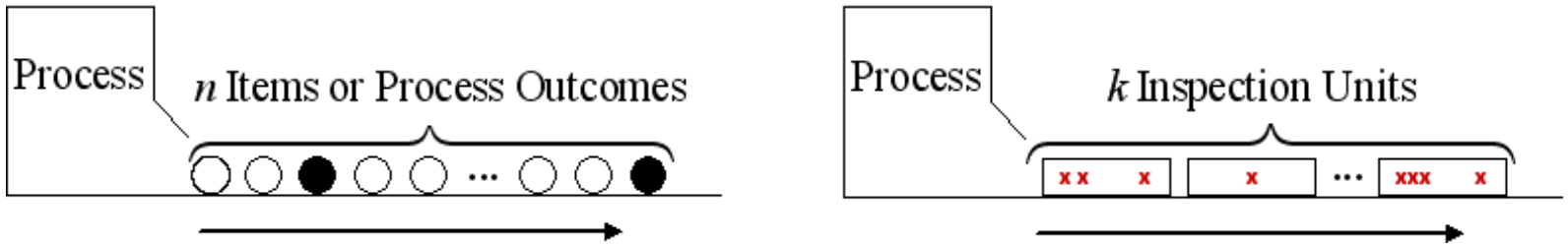

Figure 6: Comparison of "Fraction Nonconforming" and "Mean Nonconformities per Unit" Scenarios

In the p chart/"fraction nonconforming" context, n discrete items or outcomes are individually each either acceptable or nonconforming, and $\hat{p}$ is the sample fraction nonconforming. In the u chart/"mean nonconformities per unit" context, k inspection units of process output potentially suffer nonconformities, and $\hat{u}$ is the sample rate at which these are observed per unit. Both $\hat{p}$ and $\hat{u}$ are kinds of "rates," but they are intrinsically different. For one thing, it is perfectly possible for $\hat{u}$ to exceed 1.0, while $\hat{p}$ is always between 0 and 1.