Economics 603: Graduate Game Theory — Handout n+1
Iowa State University — 7 May 2009

Final Exam - Solutions

Problem 1.

1(a) [5 points]. Without loss of generality let's assume that $t_1 < t_2$. Therefore, $\min(t_1, t_2) = t_1$. The probability of a crash is therefore t_1 and driver 2 wins if there is no crash. Driver 1's payoff is therefore $-t_1$ and driver 2's payoff is $-t_1 + (1 - t_1) = 1 - 2t_1$. Driver 1 could always deviate and choose $t_1' = 0$. Therefore, only $t_1 = 0$ can be played in an asymmetric pure-strategy NE. Driver 2 certainly has no incentive to deviate since his earnings are 1 if $t_1 = 0$. However, driver 2 could deviate and choose some $t_1 = t_2 + \epsilon$ such that she just wins the race in case there is no crash. In this case her earnings are $1 - 2t_2$. We require $1 - 2t_2 \leq 0$ such that this deviation is not profitable. This will be the case for any $t_2 \geq \frac{1}{2}$. Therefore, there are infinite many NE where driver 1 chooses $t_1 = 0$ and driver 2 chooses some $t_2 \geq \frac{1}{2}$.

1(b) [5 points]. Now let's assume that both drivers randomize over the interval $[0, \bar{t}]$ according to some cdf $F(t)$ (identical for both drivers).[1] It is easy to see that the lower bound of the support has to be 0: a driver choosing the lower bound never wins and could otherwise deviate to 0 which guarantees her zero utility. We make a simple guess that the upper bound is 1.

For any $t \in [0, 1]$ players have to get exactly 0 in expectation:

$$\underbrace{\int_0^t F'(x)(-x+1-x)dx}_{\text{utility if other player chooses } x<t} - \underbrace{t(1-F(t))}_{\text{utility if other player chooses } x>t} = 0 \tag{1}$$

We can differentiate with respect to t on both sides:

$$\begin{aligned} F'(t)(1-2t) + tF'(t) - 1 + F(t) &= 0 \\ F'(t)(1-t) + F(t) &= 1 \end{aligned} \tag{2}$$

You can see that the uniform distribution over $[0, 1]$ with $F(t) = t$ and $F'(t) = 1$ solves this differential equation.

Problem 2.

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[1] This means that the probability of choosing some time $t' < t$ is $F(t)$.

2(a) [2 points]. (game tree omitted in these solutions) Player 1 has $3 \times 2 \times 2 \times 2 = 24$ strategies (4 decision nodes with three and 2/2/2 actions, respectively) and player 2 has $3 \times 3 \times 3 = 27$ strategies.

2(b) [2 points]. The following is a Nash equilibrium. Player 2 always plays C and player 1 passes 2 dollars in period 1 and plays A in period 2. This is not an SPE because the threat of playing C is not credible in period 2 - but it is Nash.

2(c) [2 points]. A unique SPE implies that the second-period subgame has a unique NE. This can only happen for $x < 0$ where (D, B) is the unique NE. Otherwise (U, A) is another NE.

2(d) [2 points]. The second period subgame can only influence first-period play if there are several NE. Therefore, we focus on the case $x \geq 0$. The best possible deviation for player 1 in period 1 is to pass 0 Dollars. The maximal "punishment" that can be imposed on player 1 is to switch between the two NE (D, B) and (U, A). Therefore, we require $|x - 1| \geq 2$. Since $x \geq 0$ this requires $x \geq 3$.

2(e) [2 points]. No. Player 1 can always guarantee herself 2 Dollars by keeping everything in period 1. In period 2, the NE either gives her $x \geq 0$ or 1.

Problem 3.

3(a) [3 points]. You have type w which is private information to you and distributed uniformly over $[0, 1]$. Your friend has type d which is private information to him and is also distributed over $[0, 1]$. Equilibrium strategies take the form of cutoff strategies – there is a critical w^* and d^* such that, for example, you go to S if $w < w^*$ and go to T otherwise.

3(b) [7 points]. Both players are indifferent between the two choices at the critical points w^* and d^*, respectively. Your friend expects you to play S with probability w^* and you expect your friend to play S with probability $1 - d^*$. Let's start with player 1:

$$\begin{aligned}
(1 - d^*)(\frac{2}{3} - w^*) + d^*(-w^*) &= d^*\frac{2}{3} \\
\frac{2}{3} - w^* - d^*\frac{2}{3} + d^*w^* - d^*w^* &= d^*\frac{2}{3} \\
\frac{4}{3}d^* &= \frac{2}{3} - w^* \qquad (3)
\end{aligned}$$

For player 2 we obtain:

$$w^*(\frac{2}{3} - w^*/2) - (1 - w^*)(1 + w^*)/2 \quad = \quad -3d^* + (1 - w^*)\frac{2}{3} \tag{4}$$

Note, that conditional on $w < w^*$ the expected value of w is $w^*/2$. Similarly, for $w > w^*$ the conditional value of w is $(1 + w^*)/2$. We can simplify the second expression to:

$$\begin{aligned} w^*\frac{2}{3} - (w^*)^2/2 - \frac{1}{2} + (w^*)^2/2 &= -3d^* + \frac{2}{3} - \frac{2}{3}w^* \\ \frac{4}{3}w^* &= \frac{7}{6} - 3d^* \end{aligned}$$

We have therefore two equations in two unknowns.

$$\begin{aligned} \frac{4}{3}w^* &= \frac{7}{6} - 3(\frac{1}{2} - \frac{3}{4}w^*) \\ \frac{2}{6} &= \frac{27 - 16}{12}w^* \\ w^* &= \frac{4}{11} \end{aligned} \tag{5}$$

We also get:

$$\begin{aligned} \frac{4}{3}d^* &= \frac{2}{3} - \frac{4}{11} \\ \frac{4}{3}d^* &= \frac{10}{33} \\ d^* &= \frac{5}{22} \end{aligned} \tag{6}$$

You and your friend go to the same place together with probability $d^*(1 - w^*) + (1 - d^*)w^*$.

Problem 4 [10 points]. There are two potential separating and two potential pooling equilibria. Let's start with the separating ones.

- *Left sender plays L and right sender plays R.* In this case, R plays U in both cases. The left type has incentive to deviate to R and get 2 instead of 1. Therefore, this is NOT a PBE.
- *Left sender plays R and right sender plays L.* In this case, R plays U if seeing R and plays D if seeing L. The left type has incentive to deviate to L and trigger R to play D in which case the sender get 3 instead of 2. Therefore, this is NOT a PBE.

Next, we check the two pooling outcomes:

- *Left and right sender play L.* In this case, the receiver is exactly indifferent between U and D which in expectation both give utility 2. Assume, the receiver plays U. Assume, that in the zero probability event of seeing R, the receiver assumes that she sees the left type. She would therefore play U. Hence, the left type would want to deviate and get 2 instead of 1. Therefore, this is NOT a PBE. Assume, that in the zero probability event of seeing R, the receiver assumes that she sees the right type. She would therefore also play U. Hence, again the left type would want to deviate. Therefore, this is NOT a PBE.

  We also consider the case where the receiver plays D when seeing L. Assume, that in the zero probability event of seeing R, the receiver assumes that she sees the left type. She would therefore play U. This gives neither the left type nor the right type an incentive to deviate to R. Therefore, this IS a PBE. Assume, that in the zero probability event of seeing R, the receiver assumes that she sees the right type. She would again play U. As before this IS a PBE.

  Hence, there is a pooling equilibrium at L. The receiver plays D if seeing L and plays U otherwise (her beliefs in the zero probability event do not affect anything).

- *Left and right sender play R.* In this case, the receiver plays U. Assume, that in the zero probability event of seeing L, the receiver assumes that she sees the left type. In this case the receiver plays U. This is a PBE. Assume, that in the zero probability event of seeing L, the receiver assumes that she sees the right type. In this case, she plays D. This would induce the right sender to deviate to L. Hence, this cannot be a PBE

To summarize there are are pooling equilibria at R and L but no separating equilibria.