

**Question 1** (100 points)

Two firms are engaged in Cournot competition. Both firms face demand  $p = \alpha - Q$  where  $p$  is the price of the good and  $Q$  is the total quantity produced by both firms. Firm 1 has constant marginal cost of production  $c$ , while firm 2 has cost of production  $\tilde{c}$  drawn from the uniform distribution over  $[0, 2c]$ . We assume  $2c < \alpha$ . Firm 2 knows its own cost  $\tilde{c}$ , while firm 1 only knows that firm 2's cost is drawn from the uniform distribution.

- (a) **(15 points)** Give a general definition of Bayesian Nash equilibrium and apply this definition to the game described above. A Bayesian Nash equilibrium of the game has the following strategies. Firm 1 chooses some output level  $q^*$  and firm 2 chooses some output level  $\tilde{q}^*(\tilde{c})$  that depends on its type  $\tilde{c}$ . Firm 1 has a single type while firm 2 has private information and therefore its strategy depends on its type.
- (b) **(15 points)** Find firm 2's best response to firm 1 playing strategy  $q$ . Show that firm 2 uses the following best response to firm 1 playing strategy  $q$ :

$$\tilde{q}(\tilde{c}) = \frac{\alpha - \tilde{c} - q}{2} \quad (1)$$

Firm 2 maximizes profits:

$$\tilde{q}(\alpha - (q + \tilde{q}) - \tilde{c}) \quad (2)$$

FOC with respect to  $\tilde{q}$  gives the following BR function:

$$\tilde{q}(\tilde{c}) = \frac{\alpha - \tilde{c} - q}{2} \quad (3)$$

- (c) **(15 points)** Find firm 1's expectation of firm 2's output computed above. Show that firm 1 has the following expectation about firm 2's output level:

$$E\tilde{q}(\tilde{c}) = \frac{\alpha - c - q}{2} \quad (4)$$

. Take the expectation of the above FOC of firm 2 and remember that  $E(\tilde{c}) = c$ .

- (d) **(15 points)** Find firm 1's best response function. Show that firm 1 has the following best-response function:

$$q = \frac{\alpha - c - E\tilde{q}(\tilde{c})}{2} \quad (5)$$

. Firm 1 maximizes profits:

$$E[q(\alpha - (q + \tilde{q}(\tilde{c})) - c)] \quad (6)$$

The FOC then becomes:

$$q = \frac{\alpha - c - E\tilde{q}(\tilde{c})}{2} \quad (7)$$

- (e) **(20 points)** Solve for the equilibrium strategies of both firms. Show that firm 1 chooses the same quantity as in a symmetric Cournot game where both firms have the same constant marginal cost  $c$ . Find the expected profits of firm 1. We substitute the expression we derived for  $E(\tilde{q}(\tilde{c}))$ . We get:

$$q^* = \frac{\alpha - c}{3} \quad (8)$$

This is the same output level as in symmetric Cournot. We can now substitute this expression into the BR function for firm 2:

$$\tilde{q}^*(\tilde{c}) = \frac{\alpha - \frac{3\tilde{c} - c}{2}}{3} \quad (9)$$

Note, that the expected output level of firm 2 is the same as for firm 1. This gives us  $\pi_1 = \frac{(\alpha - c)^2}{9}$ .

- (f) **(10 points)** Now we modify the game slightly by assuming that *the second firm* has a fixed cost  $F = \frac{(\alpha - 2c)^2}{9}$  for producing any positive amount of output (there is no fixed cost if firm 2 produces zero output). We also assume that firm 1 has no such fixed cost. Show that there is a critical cost level  $c^*$  such that for  $\tilde{c} > c^*$  the second firm will produce zero output and for  $\tilde{c} < c^*$  firm 2 will produce positive output. You do not have to show for this part that  $c^* < 2c$ . Firm 2's profit is decreasing in  $\tilde{c}$ . Therefore, as  $\tilde{c} \rightarrow \infty$  profits eventually become negative and the firm will prefer to produce zero output. Because of monotonicity the transition from positive to zero production happens at some critical level  $c^*$ .
- (g) **(10 points)** Derive an equation which defines the cutoff value  $c^*$  and show that  $c^* < 2c$  (e.g. firm 2 indeed sometimes produces zero output). Will the profits of firm 1 increase or decrease by introducing a fixed cost of production for firm 2? Can you provide an intuitive argument for the change in firm 1's profit? Let's assume that the critical cost level  $c^* > 2c$ . In this case, firm 2 would always produce and we have the same equilibrium as above. However, in this case the firm 2 type with the highest cost level,  $\tilde{c} = 2c$  makes negative profits. Using the above strategies, we find that this type of firm makes profit:

$$\frac{(\alpha - 2.5c)}{3} \left( \alpha - 2c - \left( \frac{\alpha - c}{3} + \frac{\alpha - 2.5c}{3} \right) \right) = \frac{(\alpha - 2.5c)^2}{9} \quad (10)$$

which is less than the fixed cost. Therefore, we necessarily have  $c^* < 2c$  - hence, firm 2 only produces if its cost isn't too high. The probability of production is therefore  $\frac{c^*}{2c}$ . The expected output level of firm 2 is therefore:

$$E\tilde{q}(\tilde{c}) = \frac{c^*}{2c} \frac{\alpha - \frac{c^*}{2} - q}{2} \quad (11)$$

This provides us with the following equations:

$$\begin{aligned}
 \tilde{q}^*(c^*) &= \frac{\alpha - c^* - q^*}{2} \\
 F &= \tilde{q}^*(c^*)[\alpha - c^* - (q^* + \tilde{q}^*(c^*))] \\
 q^* &= \frac{\alpha - c - \frac{c^*}{2c} \frac{\alpha - \frac{c^*}{2} - q}{2}}{2}
 \end{aligned} \tag{12}$$

The first two equations tell us that firm type  $c^*$  just makes enough profit to cover the fixed cost and is indifferent between entering and not entering. These three equations have three unknowns,  $\tilde{q}^*(c^*)$ ,  $q^*$  and  $c^*$  and therefore provide us with  $c^*$ . Firm 1's profits will *increase* because the expected output of firm 2 decreases as a result of the fixed cost which is always good news for firm 1.

**Question 2** (50 points)

- (a) **(10 points)** Each firm would have incentive to undercut the other firm slightly and almost double its profits from  $\Pi_M/2$  to  $\Pi_M$ . The Bertrand equilibrium is  $p_1 = p_2 = c$  and each firm makes zero profits.
- (b) **(25 points)** We consider the following strategies. Each firm sets price equal to the monopoly price in period 1 and every subsequent period *unless* any one of them has deviated in a previous period. In that case the firm sets price equal to marginal cost. We use the single-period deviation principle. The most profitable deviation for a firm on the equilibrium path is to undercut the rival and increase profits from  $\Pi_M/2$  to  $\Pi_M$ . This results in a gain of  $\Pi_M/2$  today but in a loss of  $\Pi_M/2$  in every future period. This is unprofitable if:

$$\Pi_M/2 < \sum_{t=1}^{\infty} \delta^t \Pi_M/2 = \frac{\delta}{1-\delta} \Pi_M/2 \tag{13}$$

This gives us  $\delta > \frac{1}{2}$ .

- (c) **(15 points)** On the equilibrium path, each of three three firm makes  $\Pi_M/3$  in each period. Deviating increases profits for one period to  $\Pi_M$ . This is unprofitable if:

$$\begin{aligned}
 \frac{2}{3} \Pi_M &< \sum_{t=1}^{\infty} \delta^t \Pi_M/3 = \frac{\delta}{1-\delta} \Pi_M/3 \\
 2 &< \frac{\delta}{1-\delta} \\
 \delta &> \frac{2}{3}
 \end{aligned} \tag{14}$$

Therefore, it is more difficult to sustain tacit collusion with three firms. This is intuitive because with three firms, each firm can gain relatively more from a single-period deviation.