

## 6.7. The sine and cosine functions

Surprisingly enough, angles and other notions of trigonometry play a significant role in the study of some biological processes. Here we review some basic facts from trigonometry. These will be applied later.

Most of the trigonometry we shall use is seen in the following diagram: The **unit circle** is the circle centered at the origin with radius 1. The distance from the point  $(x,y)$  to the origin is  $\sqrt{x^2 + y^2}$  by the Pythagorean Theorem. Since the circle has radius 1, each point has distance 1 from the origin. Hence

$$(x^2 + y^2) = 1$$

If we square each side, we see that the circle has the equation

$$x^2 + y^2 = 1.$$

Angles are measured counterclockwise from the positive x axis. Given an angle  $q$ , draw the ray (half-line) at the angle  $q$  starting at the origin. It meets the unit circle at a unique point. The most important fact about angles for this course is the following:

The point where the ray at angle  $q$  meets the unit circle is

$$(\cos(q), \sin(q)).$$

Here  $\cos(q)$  is the notation for the **cosine** of  $q$  and  $\sin(q)$  is the notation for the **sine** of  $q$ .

Thus the point on the unit circle also on the ray at angle  $q$  satisfies

$$x = \cos(q)$$

$$y = \sin(q).$$

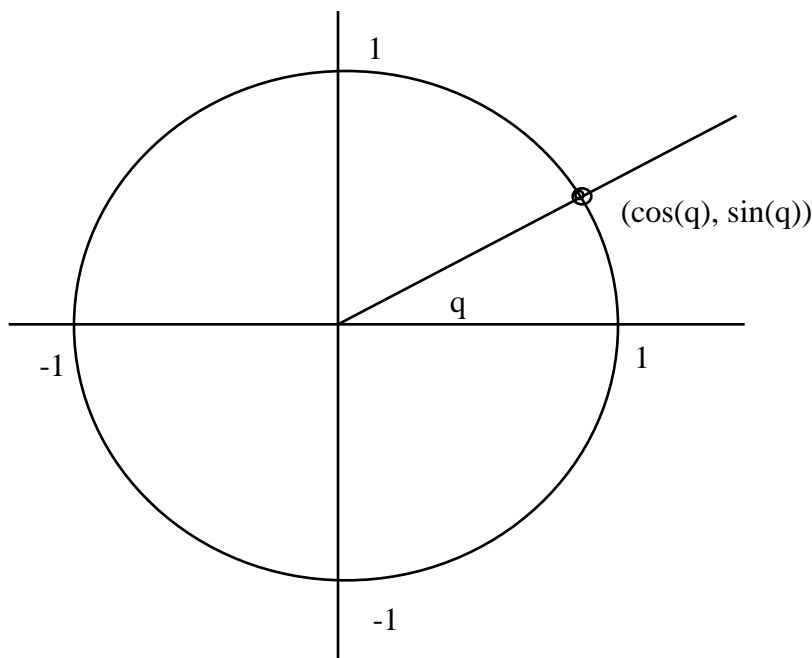


Figure 6.7.1. The unit circle and the point where the ray with angle  $q$  meets the unit circle.

To use this information, we must measure angles by some numbers. There are two common methods for measuring angles--degrees and radians. Most people start measuring angles in degrees. In this course, however, it will be important to measure angles in terms of radians instead of degrees. Thus angles will always be measured in radians.

**The angle  $\theta$  measured in radians is the length of the circular arc from the positive x axis counterclockwise until the desired angle  $\theta$  is reached.**

Since the radius of the unit circle is 1, the length of its perimeter is  $2\pi r = 2\pi(1) = 2\pi$ .

**Hence the angle for going exactly around the circle once is  $2\pi$  radians.** This corresponds to 360 degrees.

If we go only half way around the circle (corresponding to 180 degrees) then the angle is half of  $2\pi$  hence  $\pi$  radians. If we go only half of that amount (corresponding to 90 degrees) then the angle is half of  $\pi$  hence  $\pi/2$  radians. These facts let us find many cosines and sines without using the calculator.

**Example.** The angle  $\theta = 0$  radians is found by starting at the positive x-axis and going a distance 0. Hence the ray at angle  $\theta = 0$  meets the unit circle at (1,0).

Hence

$$\begin{aligned}(\cos(0), \sin(0)) &= (1,0), \\ \cos(0) &= 1, \\ \sin(0) &= 0.\end{aligned}$$

**Example.** The angle  $\theta = 2\pi$  radians is found by starting at the positive x-axis and going a distance  $2\pi$  counterclockwise around the circle. But  $2\pi$  is the total perimeter, so the relevant point is back at the starting position. Hence the ray at angle  $\theta = 2\pi$  meets the unit circle at (1,0).

Hence

$$\begin{aligned}(\cos(2\pi), \sin(2\pi)) &= (1,0), \\ \cos(2\pi) &= 1, \\ \sin(2\pi) &= 0.\end{aligned}$$

**Example.** Without a calculator, find  $\cos(\pi)$  and  $\sin(\pi)$ .

**Solution.** The angle  $\theta = \pi$  radians is found by starting at the positive x-axis and going a distance  $\pi$  counterclockwise around the circle. But  $2\pi$  is the total perimeter, so going a distance  $\pi$  arrives exactly half-way around the circle. Hence the ray at angle  $\theta = \pi$  meets the unit circle at (-1,0).

Hence

$$\begin{aligned}(\cos(\pi), \sin(\pi)) &= (-1,0), \\ \cos(\pi) &= -1, \\ \sin(\pi) &= 0.\end{aligned}$$

**Example.** Without a calculator, find  $\cos(\pi/2)$  and  $\sin(\pi/2)$ .

**Solution.** The angle  $\theta = \pi/2$  radians is found by starting at the positive x-axis and going a distance  $\pi/2$  counterclockwise around the circle. But  $2\pi$  is the total perimeter, so going a distance  $\pi/2$  arrives exactly one fourth of the way around the circle. This point is (0,1). Hence the ray at angle  $\theta = \pi/2$  meets the unit circle at (0,1).

Hence

$$\begin{aligned}(\cos(\pi/2), \sin(\pi/2)) &= (0,1), \\ \cos(\pi/2) &= 0,\end{aligned}$$

$$\sin(\pi/2) = 1.$$

**Example.** Without a calculator, find  $\cos(3\pi/2)$  and  $\sin(3\pi/2)$ .

**Solution.** The angle  $\theta = 3\pi/2$  radians is found by starting at the positive x-axis and going a distance  $3\pi/2$  counterclockwise around the circle. But  $2\pi$  is the total perimeter, so going a distance  $3\pi/2$  arrives exactly 3/4-th of the way around the circle.

This is because

$$\frac{3\pi/2}{2\pi} = \frac{3/2}{2} = \frac{3}{4}$$

This point is  $(0,-1)$ . Hence the ray at angle  $\theta = 3\pi/2$  meets the unit circle at  $(0,-1)$ .

Hence

$$(\cos(3\pi/2), \sin(3\pi/2)) = (0,-1),$$

$$\cos(3\pi/2) = 0,$$

$$\sin(3\pi/2) = -1.$$

Negative angles are found by starting at the positive x-axis and going clockwise rather than counterclockwise around the circle.

**Example.** The angle  $\theta = -\pi/2$  radians is found by starting at the positive x-axis and going a distance  $\pi/2$  clockwise around the circle. But  $2\pi$  is the total perimeter, so going a distance  $\pi/2$  arrives exactly 1/4-th of the way around the circle. This point is  $(0,-1)$ . Hence the ray at angle  $\theta = -\pi/2$  meets the unit circle at  $(0,-1)$ .

Hence

$$(\cos(-\pi/2), \sin(-\pi/2)) = (0,-1),$$

$$\cos(-\pi/2) = 0,$$

$$\sin(-\pi/2) = -1.$$

The cosine and sine of angles may be found easily on your calculator. Since we will use radians, it is important first to set the MODE of your calculator to RADIANS instead of DEGREES. In this course, all angles, unless otherwise and explicitly stated, are measured in RADIANS.

**Example.** Use your calculator to find  $\sin(\pi/6)$ .

**Solution.** Here we assume that the problem means that we are to find the sine of the angle  $\pi/6$  radians. In the calculator when we enter  $\sin(\pi/6)$ , we find that the correct value is 0.5. (If you obtain 0.009138 then your answer is wrong because your calculator is set to degree mode rather than radian mode. In fact,  $\sin(\pi/6 \text{ degrees}) = 0.009138$ , which is very different from  $\sin(\pi/6 \text{ radians})$ .)

**Example.** Use your calculator to find  $\sin(1)$  to 5 decimal places.

**Solution.** Enter  $\sin(1)$  on the calculator to find 0.84147. This is the sine of 1 radian.

**Example.** Use your calculator to find  $\cos(1)$  to 5 decimal places.

**Solution.** Enter  $\cos(1)$  on the calculator to find 0.54030. This is the cosine of 1 radian.

**Example.** Use your calculator to find the point on the unit circle at the angle 1.8 radians. Give numbers to 5 decimal places.

**Solution.** The point is  $(\cos(1.8), \sin(1.8))$ . From the calculator we find the point is  $(-0.22720, 0.97385)$ .

Figure 6.7.2 shows the graph of the function  $y = \sin(x)$ . Note how it oscillates up and down forever. This oscillation is what will be important for this course, not any

relationships about right triangles. We will use the sine function to model situations that oscillate. For example, the temperature typically oscillates, growing warmer during the day and then cooling at night. As a consequence, the level of photosynthesis in plants oscillates, increasing during the day, then waning in the evening and shutting down at night. Temperatures also show an oscillation over the year, corresponding to the seasons. Certain populations vary over the year, with bigger mosquito populations during the summer than during the winter. Oscillations also occur in modeling repetitive situations such as the beating of the heart.

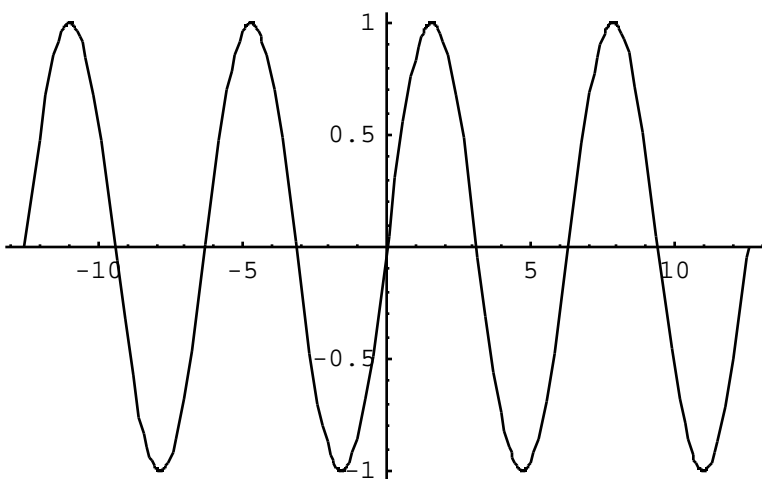


Figure 6.7.2. The graph of  $y = \sin(x)$

Figure 6.7.3 shows the graph of the function  $y = \cos(x)$ . It also oscillates up and down forever. It will also be needed to help model biological behavior that oscillates.

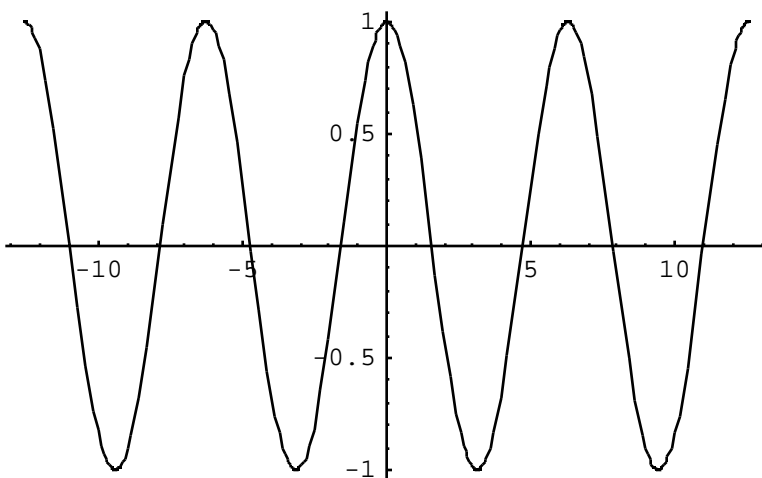


Figure 6.7.3. The graph of  $y = \cos(x)$ .

It is sometimes required to solve a trig equation. Trig equations are complicated because there are often many solutions.

**Example.** Find all solutions of  $\sin(\theta) = 0$

**Solution.** By inspecting the unit circle for when the y value or  $\sin(\theta) = 0$ , we see the solution  $\theta = 0$ . We continue along the circle and see that a second solution occurs after an angle  $\pi$ , so  $\theta = \pi$  is another solution. Continue further along the circle and we see that  $\theta = 2\pi$  is another solution. Similarly so is  $\theta = 3\pi$ ,  $\theta = 4\pi$ , etc. In fact all solutions have the form

$$\theta = n\pi \quad \text{where } n \text{ is a whole number } (0, 1, -1, 2, -2, 3, -3, \text{ etc.})$$

The solution above is a useful fact to remember.

**Example.** Find all solutions of  $\sin(5\theta) = 0$

**Solution.** From the previous example we know that all the solutions to  $\sin(x) = 0$

are  $x = n\pi$  where  $n$  is a whole number.

For the current problem,  $x$  corresponds to  $5\theta$

Hence we have  $5\theta = n\pi$

$$\theta = n\pi / 5 \quad \text{where } n \text{ is a whole number.}$$

**Example.** Find all solutions of  $\sin(5\theta - 2) = 0$

**Solution.** We know that all the solutions to

$$\sin(x) = 0$$

are  $x = n\pi$  where  $n$  is a whole number.

For the current problem,  $x$  corresponds to  $5\theta - 2$

Hence we have  $5\theta - 2 = n\pi$

$$\theta = (2 + n\pi) / 5 \quad \text{where } n \text{ is a whole number.}$$

**Example.** Find all solutions of  $\sin(5t - 2) = 0$  such that  $0 < t < 2$ .

**Solution.** From the previous example we know that all the solutions to  $\sin(5t - 2) = 0$

are  $t = (2 + n\pi) / 5$  where  $n$  is a whole number.

We now try values of  $n$  to find  $t$  satisfying  $0 < t < 2$ .

If  $n = 0$  we get  $t = 2/5 = 0.4$ .

If  $n = 1$  we get  $t = (2 + \pi) / 5 = 1.0283$ .

If  $n = 2$  we get  $t = (2 + 2\pi) / 5 = 1.65662$ .

If  $n = 3$  we get  $t = (2 + 3\pi) / 5 = 2.2850$ .

If  $n = -1$  we get  $t = (2 - \pi) / 5 = -0.2283$ .

Hence the answers satisfying  $0 < t < 2$  are

$$t = 0.4, t = 1.0283, \text{ and } t = 1.65662.$$

**Example.** Solve  $\sin(0.4x - 3) = 0$  for  $0 < x < 20$ .

**Solution.** Recall that all the solutions to

$$\sin(\theta) = 0$$

occur when  $\theta = n\pi$ .

For our problem all the solutions must therefore satisfy

$$0.4x - 3 = n\pi$$

$$x = (3 + n\pi) / 0.4$$

We try integer values of  $n$  in order to find  $x$  so  $0 < x < 20$ .

If  $n = 0$ ,  $x = 3/0.4 = 7.5$ .

If  $n = 1$ ,  $x = (3 + \pi) / 0.4 = 15.354$ .

If  $n = 2$ ,  $x = (3 + 2\pi) / 0.4 = 23.208$  which does not satisfy the constraints.

Any larger  $n$  will make the failure of the constraints worse.

If  $n = -1$ ,  $x = (3 - \pi) / 0.4 = -0.354$  which does not satisfy the constraints.

The answer is  $x = 7.5$  or  $x = 15.354$ .

**Example.** Find all solutions of  $\cos(\theta) = 0$

**Solution.** By inspecting the unit circle for when the y value or  $\cos(\theta) = 0$ , we see the solution  $\theta = \pi/2$ . We continue along the circle and see that a second solution occurs after an angle  $\pi$ , so  $\theta = \pi/2 + \pi$  is another solution. Continue further along the circle and we see that

$\theta = \pi/2 + 2\pi$  is another solution. Similarly so is  $\theta = \pi/2 + 3\pi$ ,  $\theta = \pi/2 + 4\pi$ , etc. In fact all solutions have the form

$$\theta = \pi/2 + n\pi \quad \text{where } n \text{ is a whole number.}$$

**Example.** Find all solutions of  $\cos(5t - 2) = 0$

**Solution.** From the previous example we know that all the solutions to  $\cos(x) = 0$

are  $x = \pi/2 + n\pi$  where  $n$  is a whole number.

For the current problem,  $x$  corresponds to  $5t - 2$

Hence we have  $5t - 2 = \pi/2 + n\pi$

$$t = 2/5 + \pi/10 + n\pi/5 \quad \text{where } n \text{ is a whole number.}$$

When the trig function is to have a value other than 0, the simplest methods often involve the inverse trig functions. There is a button for the inverse sine function (or  $\sin^{-1}$  function) and a button for the inverse cosine function (or  $\cos^{-1}$  function) on your calculator. Both functions should give their values in radians. Here are the basic facts underlying their use:

**Fact:**

(1) If  $y = \sin(x)$  and  $-\pi/2 \leq x \leq \pi/2$ , then

$$x = \sin^{-1}(y).$$

(2) If  $y = \cos(x)$  and  $0 \leq x \leq \pi$ , then

$$x = \cos^{-1}(y).$$

**Example.** Solve  $\sin(x) = -0.3$  if  $-\pi/2 < x < \pi/2$ .

**Solution.**  $x = \sin^{-1}(-0.3) = -0.30469$

**Example.** Solve  $\cos(x) = -0.3$  if  $0 < x < \pi$ .

**Solution.**  $x = \cos^{-1}(-0.3) = 1.87549$

**Example.** Find one solution to  $\sin(3x) = -0.5$  to 5 decimal places.

**Solution.** If  $u = \sin(v)$ , then  $v = \sin^{-1}(u)$  is one solution. Here  $u = -0.5$  while  $v = 3x$ .

Since  $\sin(3x) = -0.5$  or  $u = \sin(v)$ ,

it follows that

$$3x = \sin^{-1}(-0.5)$$

$$x = (1/3) \sin^{-1}(-0.5) = -0.17453$$

**Example.** Find one solution to  $\cos(2x - 5) = 0.25$  to 5 decimal places.

**Solution.** If  $u = \cos(v)$ , then  $v = \cos^{-1}(u)$  is one solution. Here  $u = 0.25$  while  $v = 2x - 5$ .

It follows that

$$2x - 5 = \cos^{-1}(0.25)$$

$$2x = \cos^{-1}(0.25) + 5$$

$$x = [\cos^{-1}(0.25) + 5] / 2$$

$$x = 3.15906$$

If we want to find all solutions to equations such as  $\cos(x) = -0.3$ , then we need to make use of the symmetries of the unit circle. Some of these are indicated in the next theorem:

**Theorem.**

$$\begin{aligned}\cos(-q) &= \cos(q) \\ \sin(-q) &= -\sin(q) \\ \sin(-q) &= \sin(q) \\ \cos(q+2\pi) &= \cos(q) \\ \sin(q+2\pi) &= \sin(q)\end{aligned}$$

**Proof.** Look at the unit circle. The point with angle  $q$  is  
 $(\cos(q), \sin(q))$   
 and the point with angle  $-q$  is  
 $(\cos(-q), \sin(-q))$

But clearly the points have the same  $x$  coordinate, while the  $y$  coordinates are the negatives of each other. This is seen in Figure 6.7.4, where the two points are clearly mirror images of each other.

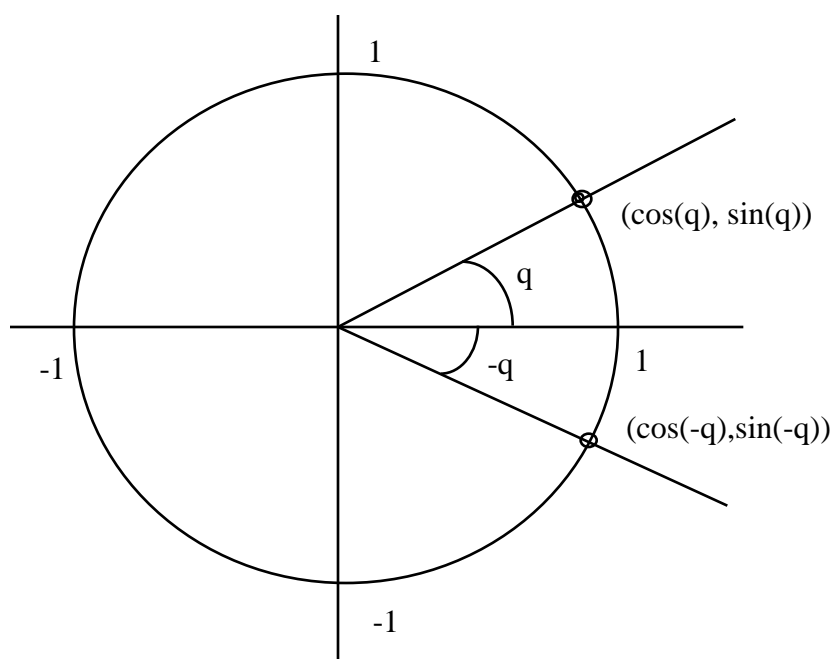


Figure 6.7.4. The points  $(\cos(q), \sin(q))$  and  $(\cos(-q), \sin(-q))$  are mirror images of each other.

The other symmetries may be seen in a similar manner.

A trigonometric equation typically has infinitely many solutions. Often, values of  $x$  are needed that are in a specific range. Once a first solution is found using the inverse trig functions, the other values are located using the symmetries in the theorem above.

These symmetries can be restated as follows:

- (1) If  $\sin(x) = y$  has one solution  $x$ , then another is  $x + 2\pi$ .
- (2) If  $\sin(x) = y$  has one solution  $x$ , then another is  $\pi - x$ .
- (3) If  $\cos(x) = y$  has one solution  $x$ , then another is  $x + 2\pi$ .

(4) If  $\cos(x) = y$  has one solution  $x$ , then another is  $-x$ .

**Example.** Solve  $\sin(x) = -0.3$  if  $0 < x < 2\pi$ .

**Solution.** One solution of the equation is  $x = \sin^{-1}(-0.3) = -0.30469$ .

This  $x$ , however, does not satisfy  $0 < x < 2\pi$ . By (1), however  $x = -0.30469 + 2\pi = 5.97849$  is a solution.

By (2), another solution is  $-\sin^{-1}(-0.30469) = 3.44629$ .

We expect exactly two answers. Hence the final answer is  $x = 5.97849$  and  $x = 3.44629$ .