5.7 Related Rates

You may think that the functions we have described are unlikely to occur in nature. We will soon see some examples. When quantities are related in complicated ways, their derivatives are also related. A "related rates" problem seeks to find the relationship among the derivatives of two or more quantities.

Example. A circular mold colony is growing in such a manner that when $t = 2$ days its radius r satisfies $r = 5$ cm and $dr/dt = 2$ cm/day. Find the rate of change of the area of the colony when $t = 2$ days.
Solution. $A = \pi r^2$ since the colony is circular.
More formally, this means that the functions A(t) and r(t) describing the area and the radius at time t are related by
$A(t) = \pi [r(t)]^2$.
By the Power Chain Rule,
$dA/dt = 2 \pi r \, dr/dt$. Hence when $t = 2$ we have
$dA/dt = 2 \pi (5) (2) = 20 \pi$ cm^2/day.

Note that in this example we needed to use a basic geometrical fact in order to find the relevant formula. Some common geometric facts include
$V = (4/3)\pi r^3$ for the volume V of a sphere of radius r,
$A = 4\pi r^2$ for the surface area A of a sphere of radius r,
$A = \pi r^2$ for the area A of a circle of radius r,
$C = 2\pi r$ for the circumference C of a circle of radius r.

In other examples, there is a more complicated formula that is given to us. The next example is purely mathematical to show the main lines of the method:

Example. Suppose u(t) satisfies $u(2) = 3$ and $u'(2) = 4$.
Let $G(t) = 2 [u(t)]^3$. Find G(2) and G'(2).
Thus in this example, we have two functions u(t) and G(t) which are related. We seek the derivative of G(t) in terms of u(t) and u '(t).
Solution.
$G(2) = 2 [u(2)]^3 = 2 [3]^3 = 54$.
$G'(t) = 2 (3) [u(t)]^2 u'(t)$ by the Power Chain Rule
Hence
$G'(2) = 2 (3) [u(2)]^2 u'(2) = 2(3) (3)^2(4) = 216$

Example. Suppose u(t) satisfies $u(2) = 3$ and $u'(2) = 4$.
Let $H(t) = 2 t [u(t)]^3$. Find H(2) and H'(2).
Solution.
$H(2) = 2 t [u(2)]^3 = 2(2) [3]^3 = 108$.
For H'(t) we must use the Product Rule:
$H'(t) = 2 t D_t[[u(t)]^3] + [u(t)]^3 D_t[2t]$
$= 2 t (3)[u(t)]^2 u'(t) + [u(t)]^3 (2)$
by the Power Chain Rule.

Hence
$H'(2) = 2(2) (3) [u(2)]^2 u'(2) + [u(2)]^3 (2)$
$= 2(2) (3) [3]^2 (4) + [3]^3 (2)$

= 486.

Example. Suppose that the population of bacteria at time t hours is P(t) mg.
It turns out in some models that
$N(t) = 0.3\, P(t) - [P(t)]^2 / 500$ mg
is the amount of a certain nutriment that must be provided to the colony to permit healthy growth. Suppose that at time t = 5 we know P(5) = 120, P'(5) = 36.
(a) Find N(5).
(b) Find N'(5).

Solution. (a) $N(5) = 0.3\, P(5) - [P(5)]^2 / 500$
$= 0.3\,(120) - [120]^2 / 500$
$= 7.2$ mg
(b) $N'(t) = D_t [\, 0.3\, P(t) - [P(t)]^2 / 500 \,]$
$= 0.3\, D_t [\, P(t)] - (1/500)\, D_t [[P(t)]^2]$
$= 0.3\, D_t [\, P(t)] - (1/500)\, D_t [[P(t)]^2]$
$= 0.3\, P'(t) - (1/500)(2)\, P(t)\, D_t[P(t)]$
using the Power Chain Rule.
$= 0.3\, P'(t) - (2/500)\, P(t)\, P'(t)$
Hence when t = 5 we know
$N'(5) = 0.3\, P'(5) - (2/500)\, P(5)\, P'(5)$
$= 0.3(36) - (2/500)\,(120)\,(36)$
$= - 6.48$ mg/hour

Example. A model for the spruce budworm population u(t) at time t involves a growth-rate correction G(t), where

$$G(t) = \frac{[u(t)]^2}{1 + [u(t)]^2}$$

Suppose u(1) = 10 and u'(1) = 0.61. Find G(1) and G'(1).

Solution:
$G(1) = 10^2 / (1 + 10^2) = 100/101.$
Note that G'(t) means $D_t[G(t)]$. Hence

$$G'(t) = \frac{[1+[u(t)]^2]\, 2\, u(t)\, u'(t) - [u(t)]^2\, (2)\, u(t)\, u'(t)}{[1+[u(t)]^2]^2}$$

Now

$$G'(1) = \frac{[1+[u(1)]^2]\, 2\, u(1)\, u'(1) - [u(1)]^2\, (2)\, u(1)\, u'(1)}{[1+[u(1)]^2]^2}$$

$$G'(1) = \frac{[1+[10]^2]\, 2\, (10)\, (0.61) - [10]^2\, (2)\, (10)\, (0.61)}{[1+[10]^2]^2}$$

$$G'(1) = \frac{(101)\ (20)\ (0.61) - (100)\ (20)\ (0.61)}{[101]^2}$$

$$G'(1) = 0.001196$$

Example. A model for the concentration x(t) of carbon dioxide in the blood at time t involves the quantity

$$V(t) = \frac{x(t)^{1.3}}{1 + x(t)^{0.3}}$$

When t = 2, x = 0.6 and x' = 0.02. Find V'(2).

Solution. By the Quotient Rule,

$$V'(t) = \frac{(1+x(t)^{0.3})\ D_t\ [\ x(t)^{1.3}] - x(t)^{1.3}\ D_t\ [1+x(t)^{0.3}]}{(1+x(t)^{0.3})^2}$$

$$= \frac{(1+x(t)^{0.3})\ (1.3)\ x(t)^{0.3} D_t\ [\ x(t)] - x(t)^{1.3}\ (0.3)\ x(t)^{-0.7} D_t\ [x(t)]}{(1+x(t)^{0.3})^2}$$

$$= \frac{(1+x(t)^{0.3})\ (1.3)\ x(t)^{0.3}(\ x'(t)) - x(t)^{1.3}\ (0.3)\ x(t)^{-0.7}(x'(t))}{(1+x(t)^{0.3})^2}$$

$$V'(2) = \frac{(1+(0.6)^{0.3})\ (1.3)\ (0.6)^{0.3}(\ 0.02) - (0.6)^{1.3}\ (0.3)\ (0.6)^{-0.7}(0.02)}{(1+(0.6)^{0.3})^2}$$

$$= 0.010726$$

Problems for 5.7

1. A circular mold colony is growing in such a manner that when t = 3 days its radius r satisfies r = 8 cm and dr/dt = 4 cm/day. Find the rate of change of the area of the colony when t = 3 days. Give your answer to the nearest whole number.
Ans. $2\pi r\ dr/dt = 2\pi(8)(4) = 201\ cm^2/day$

2. A spherical fungus is growing in such a manner that when t = 30 days its radius r satisfies r = 10 cm and dr/dt = 1.5 cm/day. Find the rate of change of the volume of the fungus when t = 30 days. Give your answer to the nearest whole number. (Hint: Recall that the volume of a sphere with radius r is $V = (4/3)\pi\ r^3$.)
Ans. $dV/dt = 1886\ cm^3/day$.

3. A bug is crawling on a wire shaped like the curve $y = 2\ x^2$, where x and y are in cm. When t = 2 sec, the bug is at x = 4 cm and is moving so that dx/dt = 3 cm/sec. Tell dy/dt.

Ans: dy/dt = 48 cm/sec.

4. A spherical balloon is inflating inside an experimental apparatus. When the radius is 7 inches, it is increasing at the rate of 1.6 in/sec. How fast is the volume of the balloon changing? Is it increasing or decreasing?
Ans: 985 in^3/sec, increasing

5. A circular mold colony is growing in such a manner that when t = 2 days its radius r satisfies r = 7 cm and dr/dt = 2 cm/day. Find the rate of change of the circumference of the colony when t = 2 days. Give your answer to the nearest tenth.
Ans. 12.6 cm/day

6. A spherical fungus is growing in such a manner that when t = 30 days its radius r satisfies r = 9 cm and dr/dt = 1.6 cm/day. Find the rate of change of the outer area of the fungus when t = 30 days. Give your answer to the nearest whole number. (Hint: Recall that the area of a sphere with radius r is $V = 4\pi r^2$.)
Ans. dA/dt = 362 cm^2/day.

7. A spherical balloon is inflating inside an experimental apparatus. When the radius is 6 inches, it is increasing at the rate of 0.35 in/sec. How fast is the area of the balloon changing? Give your answer to the nearest tenth.
Ans: 52.8 in^2/sec

8. Suppose the population P(t) satisfies that P(1) = 5 and P'(1) = 4.
Suppose $u(t) = 2 P^2(t) - P(t)$.
(a) Find u(1).
(b) Find u'(1).
Ans: (a) 45 (b) 76

9. Suppose the population P(t) satisfies that P(2) = 3 and P'(2) = 4.
Suppose $u(t) = P^2(t) / (P(t) + 1)$
(a) Find u(2).
(b) Find u'(2).
Ans: (a) 2.25 (b) 3.75

10. Suppose the population P(t) satisfies that P(1) = 9 and P'(1) = 2.
Suppose $u(t) = 2 P^2(t) + \sqrt{P(t)}$.
(a) Find u(1).
(b) Find u'(1).
Ans: (a) 165 (b) 72.3333

11. A model for an insect population P(t) at time t involves the quantity
$F(t) = 3 P(t) - 0.06 [P(t)]^2$.
Suppose P(4) = 7 and P'(4) = 6.
(a) Find F(4).
(b) Find F'(4).
Ans: (a) 18.06 (b) 12.96

12. A model for the concentration x(t) of carbon dioxide in the blood at time t involves the quantity

$$V(t) = \frac{x^{1.4}}{1 + x^{0.4}}$$

If x(3) = 0.5 and x'(3) = 0.12, find V'(3).
Ans: 0.063507

13. A model for an insect population u(t) involves the quantity
H(t) = 0.2 u(t) [1 - u(t)/50]
When t = 1, u = 6 and u' = 0.4. Find H'(1).
Ans: 0.0608

14. A model for an insect population u(t) involves the quantity

$$G(t) = 0.2\, u(t)\, [1 - u(t)/50] - \frac{[u(t)]^2}{1 + [u(t)]^2}$$

Suppose that u(1) = 17 and u'(1) = 0.3. Find G'(1).
Ans: 0.01908

15. Suppose the population P(t) satisfies that P(3) = 5 and P'(3) = 4.
Suppose $u(t) = 2\, t\, P^2(t) - P(t)$.
(a) Find u(3).
(b) Find u'(3).
Ans: (a) 145 (b) 286

16. Suppose the population P(t) satisfies that P(2) = 3 and P'(2) = 4.
Suppose $u(t) = 2\, t^2\, P^3(t)$
(a) Find u(2).
(b) Find u'(2).
Ans: (a) 216 (b) 1080

5.8 Arguments for why these more advanced quick rules are true (Optional section)

Recall the Chain Rule:
Theorem. (Chain Rule)
$(g \circ f)'(x) = g'(f(x))\, f\,'(x)$.

Why is the Chain Rule true? A complete argument is complicated, but here is the rough idea:
We wish to show that **$(g \circ f)'(x) = g'(f(x))\, f\,'(x)$.**
Since $(g \circ f)(x) = g(f(x))$, we have

$$(g \circ f)'(x) = \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h}$$

$$= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} \; \frac{f(x+h) - f(x)}{f(x+h) - f(x)}$$

[since we just multiplied by 1 in a curious form]

$$= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} \quad \frac{f(x+h) - f(x)}{h}$$

We look at these two expressions, as h goes to 0, we know that

$$\frac{f(x+h)-f(x)}{h}$$

goes to f'(x).

For the first portion of the expression, we write
$k = f(x+h) - f(x)$
so that $f(x+h) = f(x) + k$.
Now

$$\lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)}$$

$$= \lim_{h \to 0} \frac{g(f(x)+k) - g(f(x))}{k}$$

$$= \lim_{k \to 0} \frac{g(f(x)+k) - g(f(x))}{k}$$

[since as h goes to 0, we expect $k = f(x+h)-f(x)$ also to go to 0]

$$= g'(f(x))$$

since $g'(y) = \lim_{h \to 0} \frac{g(y+h)-g(y)}{h} = \lim_{k \to 0} \frac{g(y+k) - g(y)}{k}$
and this is the case where $y = f(x)$.

Hence we obtain that
$(g \circ f)'(x) = g'(f(x))\, f'(x)$
showing that the Chain Rule is true.

Theorem. (Power Chain Rule)
$D_x[\, [f(x)]^n] = n\, [f(x)]^{n-1}\, D_x[f(x)]$

We show that the Power Chain Rule arises from the general Chain Rule.

Let $g(x) = x^n$. Then
$D_x[\, [f(x)]^n] = D_x[\, g(f(x))]$
$= g'(f(x))\, f'(x)$

But $g'(x) = n\, x^{n-1}$, so
$g'(f(x)) = n\, [f(x)]^{n-1}$.
Hence $D_x[\, [f(x)]^n] = n\, [f(x)]^{n-1}\, f'(x)$
This proves the result, assuming the general chain rule.

Theorem $D_x[\, x^n] = n\, x^{n-1}$ for all constants n.

I give an argument only in the case where n is a rational number, ie when

$n = p/q$

with p and q whole numbers (integers).

Then $x^n = x^{p/q}$

So $(x^n)^q = (x^{p/q})^q = x^p$

We use the chain rule with the equation

$(x^n)^q = x^p$

The Chain Rule says:

$$(g \circ f)'(x) = g'(f(x))\, f\,'(x).$$

We think $f(x)= x^n$ and $g(x) = x^q$ (because we recognize $(x^n)^q$ as the composition of functions.) We want to find $f\,'(x)$.

But $g'(x) = q\, x^{q-1}$ since q is an integer,

so $g\,'(f(x)) = q\,(f(x))^{q-1} = q\,(x^n)^{q-1} = q\,(x^{(p/q)})^{(q-1)} = q\, x^{(p(q-1)/q)}$

$= q\, x^{(pq/q - p/q)} = q\, x^{(p - p/q)}$

$g \circ f(x) = (x^n)^q = x^p$

Since p is an integer, we have

$(g \circ f)'(x) = p\, x^{p-1}$

Hence, from the Chain Rule, we obtain

$p\, x^{p-1} = q\, x^{(p(q-1)/q)}\ f\,'(x).$

This is an equation that can be solved for f'(x):

$f\,'(x) = p\, x^{p-1} / [\ q\, x^{(p - p/q)}\ ]$

$= (p/q)\, x^{p-1-p+p/q}$

$= (p/q)\, x^{p/q-1}$

$= n\, x^{n-1}$