\[ w^{11} - 4w^1 + 2w = 0 \quad w(0) = 0 \quad w'(0) = 1 \]

\[ w = e^{rt} \]

\[ r^2 e^{rt} - 4r e^{rt} + 2e^{rt} = 0 \]

\[ r^2 - 4r + 2 = 0 \]

\[ r = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} \]

\[ r = 2 \pm \sqrt{2} \]

\[ y_1 = e^{(2+\sqrt{2})t}, \quad y_2 = e^{(2-\sqrt{2})t} \]

\[ y = C_1 e^{(2+\sqrt{2})t} + C_2 e^{(2-\sqrt{2})t} \]

\[ W = \det \begin{pmatrix} e^{(2+\sqrt{2})t} & e^{(2-\sqrt{2})t} \\ (2\sqrt{2})e^{(2+\sqrt{2})t} & (2-\sqrt{2})e^{(2-\sqrt{2})t} \end{pmatrix} \]

\[ = e^{(2+\sqrt{2})t}(2-\sqrt{2})e^{(2-\sqrt{2})t} - (2+\sqrt{2})e^{(2+\sqrt{2})t}(2-\sqrt{2})e^{(2-\sqrt{2})t} \]

\[ = 2e^{4t} \left[(2-\sqrt{2})^2 - (2+\sqrt{2})^2\right] \]

\[ = 2e^{4t} \left[4 - (2\sqrt{2})^2 - (2+\sqrt{2})^2\right] \]

\[ = 2e^{4t} \left[4 - 8 - (4 + 4\sqrt{2})\right] \]

\[ = 2e^{4t} \left[-4 - 4\sqrt{2}\right] \neq 0 \]
NON HOMOGENEOUS EQUATIONS

\( a y'' + b y' + c y = f(t) \)
WITH \( f(t) \neq 0 \)

IN THIS SECTION, WE LOOK FOR ONE SOLUTION, CALLED A PARTICULAR SOLUTION, \( y_p(t) \).

**Example:** \( y'' - 4y = e^{3t} \)
FIND A PARTICULAR SOLUTION.

METHOD IS UNDETERMINED COEFFICIENTS (JUDICIOUS GUESSING)

**GUESS:** \( y_p = A e^{3t} \)
(BECAUSE TO GET \( e^{3t} \), NEED \( e^{3t} \)).

\[
\begin{align*}
  y_p' &= 3A e^{3t} \\
  y_p'' &= 9A e^{3t} \\
  9A e^{3t} - 4A e^{3t} &= e^{3t} \\
  5A e^{3t} &= e^{3t} \\
  5A &= 1 \quad \Rightarrow \quad A = \frac{1}{5}
\end{align*}
\]

\( y_p(t) = \frac{1}{5} e^{3t} \)

**Example:** \( y'' + 2y' + y = 4t \)

**Derivatives are in the form:** \( A + B \cdot t \)

\[
\begin{align*}
  y_p' &= A \quad \Rightarrow \quad y_p'' = 0 \\
  0 + 2A + At + B &= 4t \\
  At + (2A + B) &= 4t
\end{align*}
\]

IDENTITY

COEFF OF \( t \) ON LEFT: \( A = 4 \)

CONS ON LEFT: \( 2A + B = 0 \)

\( B = -2A \equiv -2(4) = -8 \), \( y_p(t) = 4t - 8 \)
\[ y_p' = 4 \]
\[ y_p'' = 0 \]
Is \( y'' + 2y' + y = 4t \)?
Is \( 0 + 2(4) + (4t - 8) = 4t \)?
\[ 8 + 4t - 8 = 4t \? \]
\[ y_p(t) = 2te^t \]

\[ y_p(t) = Ate^t + Be^t \]
\[ y_p' = Ate^t + Ate^t + Be^t \]
\[ y_p'' = Ate^t + Ate^t + Ate^t + Be^t \]
\[ = Ate^t + 2Ate^t + Be^t \]
\[ Ate^t + 2Ate^t + Be^t + 2(Ate^t + Ae^t + Be^t) \]
\[ = Ate^t + Be^t = 2te^t \]
\[ Ate^t + 2Ate^t + Be^t + 2Ate^t + 2Be^t \]
\[ = Ate^t + Be^t = 2te^t \]
\[ (4A + 4B)e^t \]
\[ = 2te^t \]

Coeff. on \( te^t \): \[ 4A = 2 \]
\[ 4A + 4B = 0 \]

Coeff. on \( e^t \): \[ 4 + 4B = 0 \]
\[ 2 + 4B = 0 \]
\[ 4B = -2 \]
\[ B = -\frac{1}{2} \]

\[ y_p(t) = \frac{1}{2} te^t - \frac{1}{2} te^t \]
PRELIMINARY INCOMPLETE SUMMARY

To find a particular solution of

\[ ay'' + by' + cy = C t^m e^{rt} \]

guess

\[ y_p = \left( A t^m + A_{m-1} t^{m-1} + \cdots + A_0 t + A_0 \right) e^{rt} \]

Example form of the guess for

\[ y'' + 2y' + y = 4t^3 e^{2t} \]

\[ y_p = A t^3 e^{2t} + B t^2 e^{2t} + C t e^{2t} + D e^{2t} \]

\[ y'' - 4y = e^{2t} \]

So far

\[ y_p = A e^{2t} \]

\[ y''' = 2A e^{2t} \]

\[ y^{(4)} = 4A e^{2t} \]

\[ 4A e^{2t} - 4A e^{2t} = e^{2t} \]

\[ 0 = e^{2t} \]

Example:

\[ y = e^{2t} \text{ solution of } y'' - 4y = 0 \]

\[ y'' - 4y = 0 \text{ has solution } \pm 2 \]

\[ y = C_1 e^{2t} + C_2 e^{-2t} \]

So instead

\[ y_p(t) = A t e^{2t} \]

\[ y_p' = 2A t e^{2t} + A e^{2t} \]

\[ y_p'' = (2t+6)e^{2t} + 2A e^{2t} + 2A e^{2t} \]

\[ 4A e^{2t} + 4A e^{2t} - 4A e^{2t} = e^{2t} \]

\[ 4A e^{2t} = e^{2t} \]

\[ e^{2t} = 4A \]

\[ A = \frac{1}{4} \]

\[ y_p(t) = \frac{1}{4} t e^{2t} \]

There was overlap of \( f(t) \) with solution of

\[ ay'' + by' + cy = 0 \]
COMPLETE RULE
TO FIND PARTICULAR SOLUTION FOR
\( ay'' + by' + cy = Ce^rt \)

MAKE A GUESS
\( y_p = t^s(A_1t^{r_1} + A_2t^{r_2} + \ldots + A_{n-r}t^{r_{n-r}} + A_{n-r+1} + A_{n-r+2} + \ldots + A_n)e^{rt} \)

WHERE S = 0 if \( e^{rt} \) is not a solution to \( ay'' + by' + cy = 0 \) simple
S = 1 if \( e^{rt} \) is a root of the auxiliary equation for \( ay'' + by' + cy = 0 \)
S = 2 if \( e^{rt} \) is a double root of the auxiliary equation for \( ay'' + by' + cy = 0 \).

FORM OF GUESS
\( y'' - 2y' + y = 2te^t \)

HOMOGENEOUS
\( y'' - 2y' + y = 0 \)
\( r^2 - 2r + 1 = 0 \)
\( (r-1)^2 = 0 \) \( r = 1 \) is a double root

\( y_p = t^s(A_1t + B_1)et \)
S = 2
r = 1 is a double root

\( y_p = At^3e^t + Bt^2e^t \)
\( (r-1)(r+3) > 0 \)
\( c_1e^t + c_2e^{3t} \)
S = 1