CHAPTER 4
SECOND-ORDER LINEAR EQUATIONS
WITH CONSTANT COEFFICIENTS

\[ y'' - 3y' + 4y = 0 \]

Homogeneous

2nd ORDER

\[ y = f(t) \]

This has form \( ay'' + by' + cy = g(t) \)

Constant coefficients if \( a, b, c \)

are constants

It is homogeneous if \( g(t) = 0 \)

Compare

\[ y'' - 3y' + 4y = \sin t \]

Not homogeneous

SOLVING 2nd ORDER, HOMOGENEOUS EQUATIONS

STEP 1: TO SOLVE \( ay'' + by' + cy = 0 \)

MAKE THE GUESS \( y = e^{rt} \)

AND FIND \( r \).

EX:

\[ y'' - 5y' + 6y = 0 \]

\[ y = e^{rt} \]

\[ y' = re^{rt} \]

\[ y'' = r^2e^{rt} \]

\[ y'' - 5y' + 6y = r^2e^{rt} - 5re^{rt} + 6e^{rt} = 0 \]

\[ e^{rt}(r^2 - 5r + 6) = 0 \]

\[ r^2 - 5r + 6 = 0 \]

\[ r^2 = 0 \] or \[ r^2 - 5r + 6 = 0 \]

NEVER

**AUXILIARY OR CHARACTERISTIC EQUATION**
\[(r - 3)(r - 2) = 0\]
\[r - 3 = 0 \quad \text{or} \quad r - 2 = 0\]
\[r = 3 \quad \text{or} \quad r = 2\]

The solutions of form \(y = e^{rt}\) are

\[y_1(t) = e^{3t} \quad \text{and} \quad y_2(t) = e^{2t}\]

**Step 2: Use Principle of Superposition**

Theorem Suppose consider homogeneous equation \(ay'' + by' + cy = 0\).

Suppose \(y_1(t)\) and \(y_2(t)\) are solutions.

Then the "linear combination"

\[y = c_1 y_1(t) + c_2 y_2(t)\]

is also a solution.

In our example, \(y_1 = e^{3t}\) and \(y_2 = e^{2t}\)
are solutions. Hence by superposition

\[y = c_1 e^{3t} + c_2 e^{2t}\]

is a solution.

This is the general solution.

**Ex 2: Find the general solution to**

\[y'' - 9y = 0\]

**Solu:**

Guess \(y = e^{rt}\), \(y' = re^{rt}\), \(y'' = r^2 e^{rt}\)

\[r^2 e^{rt} - 9e^{rt} = 0\]

\[e^{rt}(r^2 - 9) = 0\]

\[r^2 - 9 = 0\]

\[r = \pm 3\]

Solutions \(y_1 = e^{3t}\) and \(y_2 = e^{-3t}\)

\[y = c_1 e^{3t} + c_2 e^{-3t}\]

is the general solution.
(b) Find the solution to 
\[ y'' - 9y = 0, \quad y(0) = 6, \quad y'(0) = -6. \]

**Initial conditions**

General solution \( y(t) = c_1 e^{3t} + c_2 e^{-3t} \)

\( y(0) = 6 \) \( \Rightarrow \) \( c_1 = c_1 (1) + c_2 \)

\( y'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t} \)

\( y'(0) = -6 \) \( \Rightarrow \) \( -6 = 3c_1 - 3c_2 \)

Solve \( \begin{cases} c_1 + c_2 = 6 \\ 3c_1 - 3c_2 = -6 \end{cases} \)

\( c_2 = 6 - c_1 \)

\( 3c_1 - 3(6 - c_1) = -6 \)

\( 3c_1 - 18 + 3c_1 = -6 \)

\( 6c_1 = -6 + 18 = 12 \)

\( c_1 = 2 \)

\( c_2 = 6 - 2 = 4 \)

**Answer** \( y = 2e^{3t} + 4e^{-3t} \)

**Example Solution** \( y'' + y' - 2y = 0 \)

\( y = c_1 e^t + c_2 e^{-2t} \)

or \( y = c_1 e^{-2t} + c_2 e^t \)

or \( y = k_1 e^t + k_2 e^{-2t} \)
Ex. \( y'' + 2y' + y = 0 \)

Let \( y = e^{rt} \), \( y' = re^{rt} \), \( y'' = r^2e^{rt} \)

\[ r^2e^{rt} + 2re^{rt} + e^{rt} = 0 \]

\[ e^{rt}(r^2 + 2r + 1) = 0 \]

\[ r^2 + 2r + 1 = 0 \]

\[ (r + 1)^2 = 0 \]

\[ r + 1 = 0 \text{ twice} \]

\[ r = -1 \text{ twice} \]

\[ y_1 = e^{-t} \]

In fact, \( y = te^{-t} \) is also a solution.

\[ y' = e^{-t} - te^{-t} \]

\[ y'' = -(t(-e^{-t} + e^{-t}) - e^{-t} = te^{-t} - e^{-t} - e^{-t} = te^{-t} - 2e^{-t} \]

Is \( y'' + 2y' + y = 0 \) ?

Is \( te^{-t} - 2e^{-t} + 2(-te^{-t} + e^{-t}) + te^{-t} = 0 \) ?

Is \( te^{-t} - 2e^{-t} - 2te^{-t} + 2e^{-t} + te^{-t} = 0 \) ?

Yes, \( y_2 = te^{-t} \) is also a solution.

General solution:

\[ y = c_1 e^{-t} + c_2 te^{-t} \]

Theorem: Suppose the characteristic equation of \( ay'' + by' + cy = 0 \) has a double root \( r \).

Then the general solution is

\[ y = c_1 e^{rt} + c_2 te^{rt} \]
SOLVE

\[ y'' - 4y' + 4y = 0 \]

**Ans** \[ y = c_1 e^{2t} + c_2 te^{2t} \]

SOLVE \[ y'' - 4y' + 4y = 0 \]

1. \[ y(0) = 1 \], \[ y'(0) = 5 \]
2. \[ y = c_1 e^{2t} + c_2 te^{2t} \]
3. \[ y' = 2c_1 e^{2t} + c_2 e^{2t} + c_2 te^{2t} \]
4. \[ y(0) = 1 \] so \[ c_1 + 0 = 1 \]
5. \[ y'(0) = 5 \] so \[ 2c_1 + c_2 + 0 = 5 \]
6. \[ c_1 = 1 \]
7. \[ 2c_1 + c_2 = 5 \]
8. \[ 2 + c_2 = 5 \]
9. \[ c_2 = 3 \]

\[ y = e^{2t} + 3te^{2t} \]