Introduction to Julia sets and the Mandelbrot set

Review of complex numbers:

The complex plane **C** is the set of all complex numbers x+iy where x and y are real and $i = \sqrt{-1}$. We visualize it in the usual manner: The main difference is that instead of writing (x,y) for a point we write z = x+iy. Recall the complex conjugate of x+iy is (x+iy)* = x-iy. (Usually a bar is written over x+iy rather than using an asterisk, but that requires fancier typesetting.)

Theorem. (1) $(z+w)^* = z^*+w^*$
(2) $z^{**} = z$
(3) $(zw)^* = z^*w^*$

Proof. Only the third is not obvious. But if z = x+iy and w = u+iv then
$zw = xu-yv + i\,(uy+vx)$
$z^* = x-iy$
$w^* = u-iv$
$z^*\, w^* = xu - yv -i\,(uy+vx) = (zw)^*$. QED

Let $|z| = |x+iy| = \sqrt{(x^2+y^2)}$. Then $z\, z^* = |z|$^2. Note that |x+iy| is the ordinary length of the vector (x,y). Similarly $|\, z_1 - z_2|$ is the usual Euclidean distance between the points z_1 and z_2.

Example. Find the distance between 1 + i and 3 - 5i.
Solution. We want $|\,(1+i) - (3-5i)|$.
This is $|\,-2 + 6\,i\,| = \sqrt{((-2)^2 + 6^2)} = \sqrt{40}$

Theorem. $|zw| = |z|\,|w|$.
Proof. $|zw|^2 = (zw)(zw)^* = zwz^*w^* = zz^*\, w\, w^* = |z|^2\, |w|^2$. Taking square roots, we see $|zw| = |z|\,|w|$. QED

Similarly, $|\, z / w| = |z| / |w|$ if $w \neq 0$.

Most scientific calculators allow complex numbers.

We shall find it useful to be able to compute square roots of complex numbers.
To find: $\sqrt{(a+bi)}$
Let $x + i\, y = \sqrt{(a+bi)}$
Then $(x + iy)^2 = a + bi$
$x^2 - y^2 + i\,(2xy) = a + bi$

$x^2 - y^2 = a$
$2xy = b$ so $y = b/(2x)$

$x^2 - b^2/ (4\, x^2) = a$
$4\, x^4 - b^2 = 4\, a\, x^2$
$4\, x^4 - 4\, a\, x^2 - b^2 = 0$
$x^2 = [4a \pm\sqrt{(16\, a^2 + 16\, b^2)}] / 8$
$x^2 = [4a \pm 4\sqrt{(a^2 + b^2)}] / 8$
$x^2 = [a \pm\sqrt{(a^2 + b^2)}] / 2$
But to get positive, unless b = 0, we need + sign:
$x^2 = [a +\sqrt{(a^2 + b^2)}] / 2$

$x = \pm \sqrt{[a + \sqrt{(a^2 + b^2)}]} / \sqrt{2}$
$y = \pm (b/2) \sqrt{2} / \sqrt{[a + \sqrt{(a^2 + b^2)}]}$
$y = \pm (b/\sqrt{2}) / \sqrt{[a + \sqrt{(a^2 + b^2)}]}$
Thus one solution is $\sqrt{[a + \sqrt{(a^2 + b^2)}]} / \sqrt{2} + i (b/\sqrt{2}) / \sqrt{[a + \sqrt{(a^2 + b^2)}]}$.
The other solution is its negative
$-\sqrt{[a + \sqrt{(a^2 + b^2)}]} / \sqrt{2} - i (b/\sqrt{2}) / \sqrt{[a + \sqrt{(a^2 + b^2)}]}$.

Dynamical systems on C

Suppose that f(z) is a polynomial with complex coefficients and degree $d \geq 2$. Start with a point z in **C**. Then we have the points
$f(z)$
$f^2(z) = f(f(z))$
$f^3(z) = f(f(f(z))) = f(f^2(z))$
$f^n(z) = f(f^{(n-1)}(z)))$.
The sequence $z_n = f^n(z)$ is called the **orbit** of z under f. We are interested in the behavior of $f^n(z)$ when $z \in C$.

Example. $f(z) = 2z - 1$. List the first few points in the orbit of $z_0 = 2$: $z_0 = 2$, $z_1 = f(z_0) = 3$, $z_2 = f(z_1) = 5$, $z_3 = f(z_2) = 9$, $z_4 = f(z_3) = 17$, ...
Contrast that orbit with the orbit of $z_0 = 1$. Then $z_1 = 1$, $z_2 = 1$, ...

For a while we will be primarily interested in the case where f(z) is a polynomial with complex coefficients and degree $d \geq 2$.

Here are some words to describe some behavior:
Let p be a positive integer. z is a **periodic** point with **period** p if $f^p(z) = z$ but $f^k(z) \neq z$ for any k, $0<k<p$.
A point z is **fixed** if z is periodic with period 1.
The orbit of z is **bounded** if there exists a ball B(0,r) such that $f^n(z)$ is in B(0,r) for all n.
Here $B(p,r) = \{z: |z - p| \leq r\}$ is the closed ball centered at p with radius r.
More generally, a set S is bounded if there exists a ball B(0,r) such that $S \subseteq B(0,r)$.
The orbit of z **diverges to infinity** if $\lim |f^n(z)| = \infty$.

Example. $f(z) = z^2 - 1$.
If $z = 0$, then the orbit is 0, -1, 0, -1, 0, -1, ...; we see that z=0 is a periodic point with period 2. In particular, note that the orbit of 0 is bounded.
If $z = (1+\sqrt{5})/2$, then the orbit is z, z, z, so this is a fixed point and the orbit is bounded.
If $z = -2$, the orbit is -2, 3, 8, 63, 3968, ... It looks like (and we shall prove) the orbit goes to infinity: $\lim f^n(-2) = \infty$.
$f(i) = -2$. Hence the orbit of i is i, -2, 3, 8, 63, ... Hence the orbit of z = i also diverges to infinity.
Thus different points can have very different behavior under iteration by this single function f.

Definition. If f(z) is a mapping on C, the **derivative** of f is
$f'(z) = \lim (f(z+h) - f(z))/ h$.
The same rules for differentiation apply as in Math 165 for most functions.

Ex. If $f(z) = 2z^2 + 3z + 2$, $f'(z) = 4z + 3$.
Proof: Just like for Math 165.

If $\varepsilon > 0$ and p is a point of **C** then the **ε-neighborhood** of p is defined to be $N_\varepsilon(p) = \{z \text{ in } \mathbf{C} : |z - p| < \varepsilon\}$. This is the set of points in z whose distance from p is less than ε. It consists of points inside a circle centered at p with radius ε. Another common notation is $B(p, \varepsilon)$, where the B reminds us it is round like a ball.

Definition. Let f: **C** → C and let p be a fixed point (ie, f(p) = p). We say p is a **sink** or an **attracting fixed point** if for all points x sufficiently close to p, the orbit of x is attracted to p. More precisely, p is a sink if there exists $\varepsilon > 0$ such that, for all x in $N_\varepsilon(p)$,

$$\lim_{k \to \infty} f^k(x) = p.$$

Definition. A fixed point p is a **source** or a **repelling fixed point** if all points sufficiently close to p (but not exactly at p) are repelled away from p.
More precisely a fixed point p is a source if there exists $\varepsilon > 0$ such that for all x in $N_\varepsilon(p)$, $x \neq p$, there exists $k > 0$ such that $f^k(x)$ is not in $N_\varepsilon(p)$.

Theorem. Suppose f: **C** → **C** is differentiable. Let p be a fixed point of f.
(1) If $|f'(p)| < 1$ then p is a sink.
(2) If $|f'(p)| > 1$ then p is a source.
Proof. The proof is essentially the same as for functions f: **R** → **R**.
Example. $f(z) = z^2 - 1$
$z = (1 + \sqrt{5})/2$ is fixed. But $|f'(z)| = 1+\sqrt{5} > 1$. Hence $(1 + \sqrt{5})/2$ is a repelling fixed point.

Example. $f(z) = z^2 - 2zi - 1 + i$.
Find the fixed points, tell what kind.
Soln. $z^2 - 2z\,i - 1 + i = z$

$$z^2 + z(-1-2i) + (-1+i) = 0$$
$$z = [1+2i \pm\sqrt{1 + 4i - 4 - 4(-1+i)}\,] / 2$$
$$z = [1+2i \pm\sqrt{1}\,] / 2$$
$$z = [1+2i \pm 1\,] / 2$$
$$z = 1+i \text{ or } z = i$$

$|f'(z)| = |2z-2i|$
$|f'(i)| = 0$ so i is a sink.
$|f'(1+i)| = |2+2i-2i| = 2$ so 1+i is a source.

A periodic point p of period k is called a **periodic attracting point** or **periodic sink** if is an attracting fixed point of f^k. Similarly a periodic point p of period k is called a **periodic repelling point** or **periodic source** if is a repelling fixed point of f^k.

Example. $f(z) = z^2 - 1$. Note 0 is periodic of period 2. How do we tell what kind of periodic point it is?

Theorem. If $\{p_1, p_2, ..., p_k\}$ is a periodic k orbit, then

$$(f^k)'(p1) = f'(p_1)\, f'(p_2) \ldots f'(p_k).$$

Proof. As before using the chain rule.

Theorem. Let p be a periodic point of f with period k.
(1) If $| (f^k)'(p) | < 1$ then p is a periodic sink.
(2) If $| (f^k)'(p) | > 1$ then p is a periodic source.

Completion of the example. The orbit is {0, -1}
$f'(z) = 2z$
Hence $(f^2)'(0) = [2 (0)] [2 (-1)] = 0$
This is a periodic sink.

Definition. Let $f: \mathbf{C} \to \mathbf{C}$ be a function. The **filled Julia set** F_f for f is given by $F_f = \{z \in \mathbf{C}: |f^n(z)| \text{ remains bounded for all } n\}$. This is named for Gaston Julia, 1918.

Example. For $f(z) = z^2 - 1$, $0 \in F_f$, $(1+\sqrt{5})/2 \in F_f$, $-2 \notin F_f$.

Definition. Let S be a subset of **C**. A point $x \in \mathbf{C}$ is a **boundary point** of S if for every number $\varepsilon>0$, the ball $B(x,\varepsilon)$ contains a point in S and also a point in X\S. The **boundary** of S is the set of all boundary points of S, and is written ∂S.

Example. The boundary of the square $[0,1]\times[0,1]$ is the outer perimeter. So is the boundary of $(0,1)\times(0,1)$.
The boundary of a line segment L in C is the closure of L.

Definition. The **Julia set** J_f for f is the boundary of F_f: thus $J_f = \partial F_f$.

Example. Let $f(z) = z^2$. If $|z| < 1$, then $f^n(z)$ goes to 0. If $|z| > 1$, then $f^{on}(z)$ goes to infinity. If $|z| = 1$, then $|f^n(z)| = 1$ for all n.
Hence $F_f = \{ z: |z| \le 1\}$ = the disk.
$J_f = \{z: |z| = 1\}$ = the perimeter circle.

Remark. In this easy example, the Julia set turned out to be a simple shape--a circle. It is most common, however, that the Julia set is very complicated and indeed is a fractal.

For a while we will consider different polynomials of the form $f(z) = z^2 + c$, where c is a complex or real constant. We want to learn about the Julia set.

Example. $f(z) = z^2 + i$.
Then $f(1) = 1+i$.
$f^2(1) = 3i$
$f^3(1) = -9+i$. Note $|-9+i| = \sqrt{82}$. It looks like $1 \notin F_f$, because the size of $f^{on}(1)$ seems to be growing large. But how can we be sure?

Theorem. Let $f(z) = z^2 + c$. Let $R = [1 + \sqrt{(1+4|c|)}] / 2$.
If for some $n>0$ we have $|f^n(z_0)| > R$, then $\lim |f^n(z_0)| = \infty$ and $z_0 \notin F_f$.
Example. For $f(z) = z^2 + i$, $R = (1+ \sqrt{5})/2 = 1.618$. Hence if any iterate $f^n(z_0)$ is bigger than 1.618 in absolute value, then z_0 is not in the filled Julia set. In particular, for $z_0 = 1$, we have $|f^2(1)| = 3 > 1.618$ so $f^n(1)$ goes to infinity, and $1 \notin F_f$.

The proof of the theorem requires several steps:

Lemma. $| a + b | \ge |a| - |b|$.

Proof. $|a| = |a + b - b| \le |a + b| + |-b| = |a + b| + |b|$. QED

Ex. $|z^2 + 7 + i| \ge |z^2| - \sqrt{50}$

$|z^3 + z + 17| \ge |z^3| - |z| - 17$

It is also true that $|z^3+z+17| \ge 17 - |z|^3 - |z|$ but this is unlikely to be useful since the right hand side could be negative, when the inequality really says nothing of interest.

Claim: Suppose $|z| > R$. Then $|f(z)| > |z|$.

Proof of claim:

$|f(z)| = |z^2 + c| \ge |z^2| - |c| = |z|^2 - |c|$. To obtain that $|f(z)| > |z|$, we therefore shall prove that $|z|^2 - |c| > |z|$, or $|z|^2 - |z| - |c| > 0$.

Consider the function

$g(x) = x^2 - |c| - x.$

Its graph is a parabola opening upward. Note that R is the larger root of the equation $g(x) = 0$. Hence from the graph, it follows that:

If $x > R$, then $g(x) > 0$.

Now suppose that $|z| > R$. Then $g(|z|) > 0$.

This says $|z|^2 - |c| - |z| > 0$ so $|f(z)| > |z|$.

One can then see that $|f^n(z)|$ increases forever, and in fact it will increase to infinity. To see this last, we suppose it doesn't increase to infinity. Since it is increasing, we see that there must be an $r>R$ so $|f^n(z)|$ converges to r. But since $r>R$, it follows that any point w with $|w| = r$, must have $|f(w)| > r = |w|$. By continuity it follows that if $|w|$ is near r, then $|f(w)| > r$ as well. We may choose points w of the form $w = f^n(z)$ with n so large that $|f^n(z)|$ is near r. Then $|f(w)| = |f^{(n+1)}(z)| > r$, contradicting the definition of r. This proves the theorem. QED

A set S in **C** is **bounded** if there exists $R> 0$ such that $S \subseteq B(0,R)$.

Corollary. F_f and J_f are bounded subsets of C.

Proof. Let $f(z) = z^2 + c$ and $R = [1 + \sqrt{(1+4|c|)}]/2$. If $|z| > R$, then $z \notin F_f$. Then for any number $P > R$, $F_f \subseteq B(0, P)$ and $J_f \subseteq B(0,P)$.

Corollary If $f(z) = z^2 + c$, and $z \notin F_f$ then $|f^n(z)| \to \infty$.

Corollary. $F_f = \{z: |f^n(z)| \le R \text{ for all } n\}$

Proof. Since $z \notin F_f$, it follows that $f^n(z)$ is not bounded. Hence there is some n so that $|f^n(z)| > R$, whence $f^n(z) \to \infty$. QED

Remark. This eliminates the logical possibility that the sequence $|f^n(z)|$ is not bounded but does not go to infinity because it alternates between getting huge and getting small. Thus it is not possible to have orbit behaving like 2, 6, 1, 8, 2.3, 50, 1.7, 100, -1+i, 1000, ... Hence we see that $F_f = \{z \in C: |f^n(z)| \text{ does not go to infinity}\}$.

The theorem in fact generalizes to polynomials of degree higher than 2:

Theorem. Let f(z) be a polynomial with degree m at least 2. There exists R such that $|f(z)| > |z|$ when $|z| > R$. If $|z| > R$, then $\lim |f^n(z)| = \infty$.

Proof. Write $f(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a0$ with $a_m \neq 0$. Then

$|f(z)| \ge |a_m z^m| - |a_{m-1} z^{m-1}| - ... - |a_0|$.

We seek R such that

$|a_m z^m| - |a_{m-1} z^{m-1}| - \ldots - |a_0| > |z|$ when $|z| > R$.
Let $g(x) = |a_m| x^m - |a_{m-1}| x^{m-1} - \ldots - |a_0| - x$.
We seek R so $g(x)>0$ when $x>R$. Since $a_m \neq 0$, when x is large and positive, g(x) must be positive. Let R denote the rightmost root of g(x). If $x > R$, then $g(x) >0$. Hence if $|z| >R$, then $g(|z|) >0$, so $|f(z)| >|z|$.

We now proceed as before to see that $\lim |f^n(z)| = \infty$ .QED

Drawing F_f on a computer:

Method I: the "Escape Time Algorithm":

1. Choose an R such that, if $|z| > R$, then $|f(z)| > |z|$. For example, if $f(z) = z^2 + c$, we can choose $R = [1+ \sqrt{(1+4|c|)}]/2$ or we may choose any convenient bigger value.

2. Choose a positive integer numits. [This will be the maximal number of iterations performed. The larger numits is, the slower the program.]

3. Choose a "window" W, usually of form $W = \{z = x+iy: a\leq x\leq b, c\leq y\leq d\}$. For a full picture of Ff, you should have $B(0,R) \subseteq W$ since Ff will lie inside B(0,R). Identify each point on the monitor screen with a point in the window W.

4. Choose "colors" $c_0, c_1, c_2, .., c_{numits}, c_\infty$. [These will be the colors of certain pixels.] Usually c_∞ is black.

5. For each pixel on the screen, choose a point $z \in W$ corresponding to the pixel (for example, the central point). If $|z| > R$ then color this pixel c_0;
otherwise if $|f(z)| > R$, color this pixel c_1;
otherwise, if $|f^2(z)| > R$, color the pixel c_2;
otherwise, if $|f^3(z)| > R$, color the pixel c_3;
In general, for the first n so $|f^n(z)|>R$ we color the pixel color c_n; if none of $|f^n(z)| > R$ for n = 1, 2, ..., numits, then color the pixel c_∞.

Thus, for a black and white picture where the black points correspond to F_f, we choose $c_0 = c_1 = c_2 = \ldots = c_{numits}$ = white. Points will be colored white if you know that their iterates are going to infinity.

Choose c_∞ = black. Thus every selected point which was actually in F_f should be colored black. The gradations in color are in the complement of F_f.

We would interpret the black points as a picture of F_f and white points as points not in F_f. Then J_f will be interpreted as the boundary of the set of black points. One could use an edge detecting program to draw J_f once one has F_f drawn.

While we say that the black points are a picture of F_f, actually that is a guess. We do not know that each black pixel contains a point of F_f because maybe the points iterate to infinity but it requires (2 numits) iterations before the value gets bigger than R.

We do not know that each white pixel contains no points of Ff because we only tested one point in each such pixel. Since each pixel actually corresponds to many points in the complex plane, we expect that often some of those points are in Ff and others are not; our picture gives no such information.

Moreover, there are likely to be round-off errors in the calculations.

Hence our faith in the pictures is somewhat conjectural. On the other hand, repeated production of the pictures on many computers by many people seem to lead to very similar pictures, even when the resolution and computer time are increased. This gives us some experimental confidence in the pictures.

Comments:

1. When the number of pixels is large, in order to get reasonable detail, you must use larger values for numits. [Thus for z^2 -1.1, we see that numits = 17 is much better than numits = 7.] This slows the process.
3. All points in F_f are the same color.
4. Color resolution is possible only for points outside F_f.
5. When the window is small, high precision representation of the numbers is required, and numits must be large to get resolution.
6. Roundoff errors can lead to the wrong decision at a pixel.

The Mandelbrot Set

Informal Definition. The set $A \subseteq C$ is **connected** if it is all in one piece; it is **disconnected** if it has more than one piece.

With more precision, the subset A of the complex plane is **disconnected** if there are open sets U and V of C which are nonempty and disjoint such that A is contained in $U \cup V$, U has a point of A, and V has a point of A. It is **connected** if it is not disconnected.

[There is a more technical definition needed for careful arguments. The formal definition is as follows: The closed set A is **disconnected** if A can be written as $A = U \cup V$, where U and V are both closed and nonempty, and $U \cap V = \emptyset$. A is **connected** if it is not disconnected.]

Example. The square is connected; so is a square with a whisker. Two disjoint squares are disconnected, even if one is inside the other. The Cantor set is disconnected. Connectedness is a fundamental property of a set.

In general, a compact set A in the plane is connected if it cannot be separated into two nonempty compact sets B and C with no points in common (and therefore a positive distance apart).

We now shall change our notation to emphasize the role of the constant c. We shall write

$f_c(z) = z^2 + c \quad$ for $c \in C$.

Thus each $c \in C$ gives a different function f_c.

Example. $f_2(3) = 11$, $f_3(2) = 7$.

When we compute F_{f_c}, we locate all z so the sequence $f_c^n(z)$ does not go to infinity. Note that c is fixed throughout the calculation.

Definition. The **Mandelbrot set** M(2) (for the family of functions $f_c(z) = z^2+c$) is

$M(2) = \{c \in C\text{: } F_{f_c} \text{ is connected}\}$

It turns out to equal $\{c \in C\text{: } J_{f_c} \text{ is connected}\}$.

Example. $0 \in M(2)$ since $f_0(z) = z^2$ and F_{f_0} is the unit disc, which is connected.

Example. From the pictures, $.11031-.67037\, i \notin M(2)$, but $i \in M(2)$.

Note that M(2) is a subset of parameter space {c}. Here is the picture of M(2). Each point of M corresponds to a point c such that the Julia set of fc is connected. The x-axis is a horizontal line of symmetry. The cusp at the right of the cardioid is 0.25, and the cardioid joins a large circular region on the left portion of the x-axis at -0.75. Note that the origin is off-center. Locate i above the origin.

Remark. One can similarly define other Mandelbrot sets. For example, if we consider the family of functions
$g_c(z) = z^3 + c$, then we can define another Mandelbrot set

$M(3) = \{c \in C: F_{g_c}$ is connected$\}$.

Or if $h_c(z) = z^3 + z + c$, then we can define another Mandelbrot set

$M = \{c \in C: F_{h_c}$ is connected$\}$.

We can even get more complicated Mandelbrot sets with more than one parameter:
if $h_{cd}(z) = z^3 + cz + d$, then we can consider a Mandelbrot set

$M = \{(c,d) \in C^2: F_{h_{cd}}$ is connected$\}$. This is, of course, hard to draw since its picture lies in C^2, which is like R^4.

In order to draw a Mandelbrot set M, we need a criterion.

Definition. A **critical point** of f is a point $z_0 \in C$ such that $f'(z_0) = 0$.

Recall: The **basin of infinity**, $A(\infty)$, for f is $\{z: |f^n(z)|$ goes to $\infty\}$.

Thus $F_f \cup A(\infty) = C$.

Theorem. Suppose that f(z) is a polynomial of degree at least 2. Then J_f and F_f are connected if and only if no critical point lies in $A(\infty)$.

Example. If $f(z) = z^2 + 1$, is J_f connected?

Solution. The only critical point is 0. But if we iterate 0, we get the orbit 0, 1, 2, 5, 26,... where the critical radius $R = (1+\sqrt{5})/2$. Hence $0 \in A(\infty)$ and J_f is not connected. Hence $1 \notin M(2)$.

Example. Is 0 in M(2)? If $f(z) = z^2$, then F_f is the unit disk, hence is connected. Thus 0 does lie in M(2). But we can also see it because the only critical point is z = 0, which is fixed, hence is not in $A(\infty)$.

Example. Is -1 in M(2)? Here $f(z) = z^2-1$. The only critical point is 0 and the orbit of 0 is 0, -1, 0, -1, ... so 0 is not in the basin of infinity. Then -1 lies in M(2).

Example. Let $f(z) = z^3 - 3z$. Is J_f connected?

Solution. The critical points are 1 and -1. The orbit of 1 is 1, -2, -2, -2, ... while the orbit of -1 is -1, 2, 2, 2, ... Hence neither is in the basin of infinity, so J_f is connected.

This criterion lets us produce an algorithm for drawing M. Essentially we can use the escape time algorithm:

1. Each pixel corresponds to some collection of complex numbers; for each pixel choose c one such point.

2. Compute R_c corresponding to f_c:
$R_c = [1+\sqrt{(1+4|c|)}]/2$.

3. Compute $f_c^n(0)$ for n = 1, 2, ..., numits. If any of these has absolute value greater than R_c, then we know that the orbit of 0 goes to infinity, hence J_{f_c} is not connected, hence $c \notin M$. Color the pixel appropriately.

If all of these have absolute value less than R_c, then we assume that the orbit of 0 remains bounded, so we assume $c \in M$.

The method has the same advantages and disadvantages of the escape time algorithm for drawing Julia sets.

Geometry of the Mandelbrot set M(2)

Theorem. M(2) is bounded. Indeed, if $|c| > 2$ then c is not in M(2).

Proof. Since $f(z) = z^2 + c$, the critical point is 0. The escape radius is
$R = (1+\sqrt{(1+4|c|)})/2$.
Then c will fail to be in M(2) if f(0) lies outside the escape radius:
$|f(0)| > R$
$|c| > (1+\sqrt{(1+4|c|)})/2$
$2|c| > 1+\sqrt{(1+4|c|)}$
$2|c| - 1 > \sqrt{(1+4|c|)}$
$(2|c| - 1)^2 > 1+4|c|$
$4|c|^2 - 4|c| + 1 > 1 + 4|c|$
$4|c|^2 - 8|c| > 0$
$4|c| (|c| - 2) > 0$
Hence this will be true when $|c|-2 > 0$
or $|c|>2$. Thus if $|c|>2$, then c is not in M(2).
QED

Corollary. $M(2) \subseteq \{c \text{ in } C: |c| \leq 2\}$.

Let p be a fixed point. The **basin of attraction** for p is

$A(p) = \{z: \lim f^k(z) = p\}$.

If p is a sink, then A(p) includes $N_\varepsilon(p)$ for some $\varepsilon > 0$.

More generally, let p be a periodic point of period k. Let the orbit of p be $O(p) = \{p, f(p), ..., f^{k-1}(p)\}$. The **basin of attraction** for p is

$A(p) = \{z: \lim f^{kn}(z) \text{ is in } O(p)\}$

$= \{z : \lim f^{kn}(z) = p\} \cup \{z : \lim f^{kn}(z) = f(p)\} \cup \{z : \lim f^{kn}(z) = f^2(p)\} \cup \ ...$

$\cup \{z : \lim f^{kn}(z) = f^{k-1}(p)\}$

If p is a periodic sink, then A(p) includes $N_\varepsilon(p)$ for some $\varepsilon > 0$.
A(p) includes all points that get closer and closer periodically to the members of O(p). So A(p) might also be called the basin of attraction of the orbit O(p).

If $\lim f^{kn}(z) = p$, then $\lim f^{kn+1}(z) = f(p)$.

Theorem. Suppose f is a complex polynomial and p is an attracting periodic k.. Then there exists a critical point z_0 of f such that $z_0 \in A(p)$.

Cor. Suppose $f(z) = z^2 + c$. Suppose f has a periodic point p that is a sink. Then $c \in M(2)$.
Proof. The only critical point of f is $z = 0$. Since p is a sink, $0 \in A(p)$. Hence $0 \notin A(\infty)$, whence $c \in M(2)$.
QED

Cor. Suppose $f(z)$ is a polynomial of degree m, where $m \geq 2$. Then there are at most (m-1) attracting periodic orbits.
Proof. The critical points satisfy $f'(z) = 0$, which is a polynomial equation of degree (m-1), which can have at most (m-1) roots. QED

For example, if $f(z) = z^2 + c$ then there might be an attracting periodic point p of period 25, but the members of O(p) are the only attracting periodic points of f. Other periodic points are not attracting hence are repelling or neither.

This theorem lets us identify parts of M(2) by the existence of certain kinds of sinks.

Question: For which c does $f(z) = z^2 + c$ have a fixed point z that is a sink because $|f'(z)| < 1$? In that case there is z such that $z^2 + c = z$

$$|2z| < 1$$

Hence the boundary consists of all z such that $|z| = 1/2$ and

$$z^2 - z + c = 0$$

$$c = z - z^2 \quad \text{when } |z| = 1/2.$$

We will draw this using computer or calculator. To do so, note
$z = x + i\, y = (1/2) \cos(t) + i\, (1/2) \sin(t)$
where $x = (1/2) \cos(t)$
and $y = (1/2) \sin(t)$ and $0 \leq t \leq 2\pi$.
Hence $z^2 = x^2 - y^2 + i\, 2xy$
$c = x - x^2 + y^2 + i\, (y - 2\, x\, y)$

Thus $c = c_1 + i\, c_2$ where
$c_1 = x - x^2 + y^2 = (1/2) \cos t - (1/4) \cos^2(t) + (1/4) \sin^2(t)$
$c_2 = y - 2\, xy = (1/2) \sin t - (1/2) \cos(t) \sin(t)$

We plot this for $0 \leq t \leq 2\pi$.

We see that the big cardioid of M(2) corresponds to the c so f_c has a fixed point sink. Similarly, the left disk centered at -1 with radius 1/4 corresponds to c so f_c has a periodic 2 sink. If you look for c so f_c has a periodic 3 sink, you get the prominent disks centered at $0.1226 \pm i\, 0.7449$ as well as a tiny bud centered at $c = -1.7549$.