Review for final exam in Math 365.

Example. Find all values of cube root (1+i).
Soln. Write $1 + i = \sqrt{2}\exp(i\pi/4 + 2\pi ik)$
Hence the cube root is $[\sqrt{2}\exp(i\pi/4 + 2\pi ik)]^{(1/3)}$
$= 2^{(1/6)}\exp(i\pi/12 + 2\pi ik/3)$

Example. Simplify $e^{(2+3i)}$

Example. Is $\phi = y^2 + x^2 - 2xy$ harmonic?
$f_x = 2x-2y$
$f_{xx}=2$
$f_y=2y-2x$
$f_{yy}=2$
$f_{xx}+f_{yy} = 4$. Hence no.

Example. Let $u(x,y) = x^3 - 2x^2 - 3xy^2 + 2y^2$. Find all conjugate harmonic functions of u.
Ans… $v(x,y) = 3x^2y - 4xy - y^3 + K$

Example. If $f(z) = 2z / [(z-1)^2 (z^2+2)]$, find Res(1) and Res($i\sqrt{2}$).
Solution. The multiplicity is 2 hence we look at

$$2z / (z^2 + 2)$$

Its derivative is

$$\frac{(z^2+2)(2) - 2z(2z)}{(z^2+2)^2}$$

with limit $[(3)(2)-4]/9 = 2/9$

Example. Find the domain of analyticity for Log(z + 1 - i). Tell its branch cut. Tell its derivative. Tell its value when z = -2.

Example. (Wedges) Find a function f(x,y) that is harmonic in the wedge between $\theta = 2\pi/3$ and $\theta = 4\pi/3$. It must take the value 3 on the top and 5 on the bottom.

Solution. We need a harmonic function with these level curves.
$f(x,y) = A\,\arg0(z) + B$. Now
$3 = A(2\pi/3) + B$
$5 = A(4\pi/3) + B$
Subtract: $2 = A(2\pi/3)$
$A = 3/\pi$
$B = 3 - 2 = 1$
$f(x,y) = (3/\pi)\arg0(z) + 1$

Example. Find $\int_\gamma \cos z\,dz$ where g is the circle centered at 0 traversed clockwise from $-\pi$ to πi.
Solution. An antiderivative is $F(z) = \sin z$.
Hence the integral is $F(\pi i)-F(-\pi) = \sin(\pi i) - \sin(-\pi)$
$= (e^{-\pi} - e^{\pi})/(2i)$

Example. Int_Gamma [(5z-2) / (z^2 - z) dz]
where Gamma is a curve containing 1 and 0 inside.
Soln. Use residues. Ans 14πi
(avoid partial fractions)

Example.

$$I = \int_\Gamma \frac{e^{2z}}{(z-1)^2}\, dz$$ if Γ goes around 1 positively.

Soln: Here $f(z) = e^{2z}$ so $f'(z) = 2 e^{2z}$
The answer is $2\pi i f'(1) = 2\pi i\, 2 e^2 = 4\pi i e^2$

Example. Find the best K you can such that
$\left| \int_g e^z/(z-1)\, dz \right| \leq K$
where g is the line segment from 3-2i to 3+2i

Solution. $| e^z / (z-1)|$
$= |e^z| / |z-1|$

But z = 3-2i+t(4i) = 3+i(4t-2)
$e^z = e^{3+i(4t-2)}$
so its absolute is $\leq e^3$
$|z-1| \geq 2$
Hence $1 / |z-1| \leq 1/2$
Hence
$|e^z| / |z-1| \leq e^3 (1/2)$
The length of the curve is 4
Hence the integral is $\leq e^3 (1/2) (4) = 2 e^3$

Example. If $f(z) = \sum j(z-3)^j / 2^j$ for $|z-3| < 2$
Find Int $f(z)/(z-3)^3$ dz over $|z-3| = 1/2$.
Ans: By looking at series it is
$2\pi i f''(3)/ 2! = \pi i$

Example. Tell the types of each isolated singularity in

$$f(z) = \frac{(z-1)^2 z^2 (e^z - 1)}{z^5 (z-2)^5 (z-1)}$$

1: removable; 0: pole of order 2; 2: pole of order 5

Work an integral from Chapter 6.2, 6.3