

11.2 Partial derivatives

When we deal with functions of more than one variable, we have to modify the tools of calculus. In this section we will see the modification needed for derivatives.

Recall if $f(x)$ is a function, then the **derivative** of $y = f(x)$ is defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Thus if x_0 is a particular value for x , then the derivative at x_0 is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Suppose $z = f(x,y)$ is a function of the two variables x and y . If y is held constant at $y = y_0$, then $f(x,y_0)$ is a function of a single variable x . Its derivative at $x = x_0$ is called the **partial derivative of f with respect to x** at (x_0, y_0) .

The usual notation is written with curly d's as follows:

$$f'_x(x_0, y_0)$$

or

$$\frac{df}{dx}(x_0, y_0)$$

It is defined by the formula

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Note in the formula that only x changes; y remains fixed at y_0 .

The computation is really quite simple. To find f'_x , think of y as constant and use the normal rules for differentiating with the variable x .

Example. Let $f(x,y) = 4x^3y^2$.

To find $f'_x(x,y)$ we think of y as fixed or constant and only x varying; then we compute the derivative. Thus we think of this expression as Cx^3 where $C = 4y^2$ is temporarily regarded as constant since it contains no x . The derivative is then $C3x^2$.

Hence $f'_x(x,y) = 12x^2y^2$.

We evaluate the partial derivative at the point $(1,2)$ by replacing x by 1 and y by 2. Thus

$$f'_x(1,2) = 12(1)^2(2)^2 = 48.$$

Example. $f(x,y) = 3e^{(4xy)}$. Then $f'_x(x,y) = 12ye^{(4xy)}$.

Similarly, suppose x is held constant at $x = x_0$; then $f(x_0,y)$ is a function of a single variable y . Its derivative at $y = y_0$ is called

the **partial derivative of f with respect to y** at (x_0, y_0) .

The usual notation is written with curly d's:

$$df / dy (x_0, y_0)$$

or

$$\frac{f'(x_0, y_0)}{y}$$

It is defined by the formula

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

In this formula only y changes; x remains fixed at x_0 .

As in the case of the partial derivative with respect to x , to find f / y , you should think of x as constant and use the normal rules for differentiating with variable y .

Example. Let $f(x, y) = 4x^3y^2$.

To find $f / y(x, y)$ we regard the expression as Cy^2 where $C = 4x^3$ is temporarily regarded as constant since it contains no y .

Then $f / y(x, y) = 8x^3y$.

At the point $(1, 2)$, the value is

$$f / y(1, 2) = 8(1)^3(2) = 16.$$

Example. $f(x, y) = 3e^{(4xy)}$. Then $f / y(x, y) = 12xe^{(4xy)}$.

Example. $f(x, y) = x^2 \cos(2xy^2)$. Find f / x and f / y .

Solution. For f / x we differentiate thinking of y as constant. We still need to use the product rule. Hence

$$\begin{aligned} f / x &= x^2 / x [\cos(2xy^2)] + \cos(2xy^2) / x [x^2] \\ &\quad \text{by the product rule} \\ &= x^2 (-\sin(2xy^2)) / x (2xy^2) + \cos(2xy^2) (2x) \\ &\quad \text{by the chain rule for differentiating the sine function} \\ &= x^2 (-\sin(2xy^2)) (2y^2) + \cos(2xy^2) (2x) \\ &= -2x^2y^2 \sin(2xy^2) + 2x \cos(2xy^2) \end{aligned}$$

$$\begin{aligned} f / y &= x^2 / y [\cos(2xy^2)] \\ &\quad \text{since } x^2 \text{ is constant as far as } y \text{ is concerned} \\ &= x^2 (-\sin(2xy^2)) / y (2xy^2) \\ &= x^2 (-\sin(2xy^2)) (4xy) \\ &= -4x^3y \sin(2xy^2) \end{aligned}$$

If $z = f(x, y)$, then $f(x, y) / x$ may also be written z / x .

Example. $z = x / (2x + 3y)$. Find $z / y(1, 2)$.

Solution. Since the expression is a fraction, we use the quotient rule.

$$\begin{aligned} z/ y(x,y) &= \frac{(2x + 3y) / y(x) - x / y(2x + 3y)}{(2x + 3y)^2} \\ &= \frac{-3x}{(2x + 3y)^2} \end{aligned}$$

How should we interpret these partial derivatives? Just as the ordinary derivative has different interpretations in different contexts, so does a partial derivative.

Case (1) If $z = f(x,t)$ is the height of a string at position x and time t , then z/ t is the instantaneous rate of change of the height with respect to time (where the position x remains constant). This is the vertical velocity of the string at position x .

Example. A vibrating string has height $z = f(x,t) = 3 \sin(x/4) \cos(t/4)$ cm at position x cm along the string at time t seconds. Find the vertical velocity where $x = 1$ at time $t = 2$.

Solution. We want $f/ t(1, 2)$.
 But $f/ t(x,t) = 3 \sin(x/4) / t(\cos(t/4))$
 $= 3 \sin(x/4) (-\sin(t/4)) / t(t/4)$
 $= 3 \sin(x/4) (-\sin(t/4)) (-1/4)$
 $= -(3/4) \sin(x/4) \sin(t/4)$

Hence $f/ t(1,2) = -(3/4) \sin(1/4) \sin(2/4)$
 $= -(3/4) \sin(1/4) \sin(1/2)$
 $= -(3/4) \sin(1/4) \sin(1/2)$
 $= -1.66608 \text{ cm/sec}$

Note that because f is in units of cm and t is in units of sec, then f/ t has units cm/sec.

Case (2) If $T = f(x,y)$ is the temperature at the location (x,y) , then $f/ x(x,y)$ is the rate of change of temperature as x changes but y is held constant. This is called the **rate of change of T with respect to x** since the variable x is changing. It tells the rate of change of the temperature as a point moves horizontally (in the x direction). Similarly $f/ y(x,y)$ is the rate of change of temperature as y changes but x is held constant. It is called the **rate of change of temperature with respect to y**, since y is the variable that changes. It tells how fast the temperature changes as a point moves vertically (in the y direction).

Example. The temperature at point (x,y) is $z = \cos(3x+2y)$ degrees, where x and y are in feet.
 (a) Find the rate of change of temperature with respect to x when y is constantly 2, at $x = 1$.
 (b) Find the rate of change of temperature with respect to y when x is constantly 1, at $y = 2$.

Solution.
 (a) We want $z/ x(1,2)$
 But $z/ x = -3 \sin(3x+2y)$
 so $z/ x(1,2) = -3 \sin(7)$ degrees/ft.

This means that for each foot you move in the positive x -direction from $(1,2)$, the temperature decreases by $3 \sin(7)$ degrees.

(b) We want $z/ y (1,2)$
 But $z/ y = -2 \sin(3x+2y)$
 so $z/ y (1,2) = -2 \sin(7)$ degrees/ft.

Case (3) Tangent planes

If $z = f(x,y)$ is a function, then its graph is a surface in three dimensions. A surface typically has a tangent plane at a point, analogous to the tangent line at a point of a curve in the xy plane. The tangent plane meets the surface at the given point and just fits the surface.

The equation of the tangent plane to $z = f(x,y)$ at the point where $x = x_0$, $y = y_0$ is given by

$$z = z_0 + f/ x (x_0, y_0) (x - x_0) + f/ y(x_0, y_0) (y - y_0)$$

where $z_0 = f(x_0, y_0)$.

This equation is analogous to the equation of the tangent line to a curve $y = f(x)$ in the plane at the point where $x = x_0$. In that case we let $y_0 = f(x_0)$ and the equation of the tangent line can be written

$$y = y_0 + f'(x_0) (x - x_0).$$

Example. Find the equation of the tangent plane to

$z = f(x,y) = x^2 + 3y^2 + xy$
 at the point where $x = 1$ and $y = 2$.

Solution.

Here $x_0 = 1$ and $y_0 = 2$. Hence $z_0 = f(1,2) = 15$.

$f/ x(x,y) = 2x + y$ so $f/ x(1,2) = 4$.

$f/ y(x,y) = 6y + x$ so $f/ y(1,2) = 13$.

The equation of the tangent plane is

$$z = 15 + 4(x - 1) + 13(y - 2)$$

or

$$z = 4x + 13y - 15$$

Second partial derivatives

Just as we can differentiate a function of one variable more than once, we can also take partial derivatives more than once. If $z = f(x,y)$ is a function of two variables, there are four possible kinds of second partial derivatives:

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} [f/ x]$ means the result of taking the partial derivative

with respect to x of the partial derivative with respect to x .

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} [f/ x]$ means the result of taking the partial derivative

with respect to y of the partial derivative with respect to x .

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] \text{ means the result of taking the partial derivative}$$

with respect to x of the partial derivative with respect to y .

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial y} \right] \text{ means the result of taking the partial derivative}$$

with respect to y of the partial derivative with respect to y .

The interpretations of these second order partial derivatives are subtle, but they are sometimes needed in formulas.

Example. Let $f(x,y) = x^5 y$

$$\frac{\partial f}{\partial x} = 5 x^4 y.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (5 x^4 y) = 20 x^3 y.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (5 x^4 y) = 5 x^4.$$

$$\frac{\partial f}{\partial y} = x^5.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (x^5) = 0.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (x^5) = 5 x^4.$$

Note in this example that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. This is no mere coincidence. In fact, we have the following result. For most functions $f(x,y)$ that we deal with on a daily basis,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Similar considerations apply to functions of more than two variables as well:

Example. Suppose $T(x,y,z) = x^2 y^3 z^4$ is the temperature at (x,y,z) . then

$$T/\ x = 2 x y^3 z^4$$

$$T/\ y = 3 x^2 y^2 z^4$$

$$T/\ z = 4 x^2 y^3 z^3$$

$$\frac{\partial^2 T}{\partial x^2} = \left(\frac{\partial}{\partial x} \right) \left(T/\ x \right) = \left(\frac{\partial}{\partial x} \right) \left(2 x y^3 z^4 \right) = 2 y^3 z^4$$

$$\frac{\partial^2 T}{\partial y^2} = \left(\frac{\partial}{\partial y} \right) \left(T/\ y \right) = \left(\frac{\partial}{\partial y} \right) \left(3 x^2 y^2 z^4 \right) = 6 x^2 y z^4$$

$$\frac{\partial^2 T}{\partial z^2} = \left(\frac{\partial}{\partial z} \right) \left(T/\ z \right) = \left(\frac{\partial}{\partial z} \right) \left(4 x^2 y^3 z^3 \right) = 12 x^2 y^3 z^2$$

$$\frac{\partial^2 T}{\partial x \partial y} = \left(\frac{\partial}{\partial x} \right) \left(T/\ y \right) = \left(\frac{\partial}{\partial x} \right) \left(3 x^2 y^2 z^4 \right) = 6 x y^2 z^4$$

$$\frac{\partial^2 T}{\partial x \partial z} = \left(\frac{\partial}{\partial x} \right) \left(T/\ z \right) = \left(\frac{\partial}{\partial x} \right) \left(4 x^2 y^3 z^3 \right) = 8 x y^3 z^3$$

$$\frac{\partial^2 T}{\partial y \partial z} = \left(\frac{\partial}{\partial y} \right) \left(T/\ z \right) = \left(\frac{\partial}{\partial y} \right) \left(4 x^2 y^3 z^3 \right) = 12 x^2 y^2 z^3$$

$$\frac{\partial^2 T}{\partial y \partial x} = \frac{\partial^2 T}{\partial x \partial y} = 6 x y^2 z^4$$

$$\frac{\partial^2 T}{\partial z \partial x} = \frac{\partial^2 T}{\partial x \partial z} = 8 x y^3 z^3$$

$$\frac{\partial^2 T}{\partial z \partial y} = \frac{\partial^2 T}{\partial y \partial z} = 12 x^2 y^2 z^3$$