Chapter 11. Multivariable Calculus.

11.1 Functions of more than one variable

A real-valued function of 1 variable is a rule f that assigns to each number x in some set D of the number line a real number f(x). Usually we give the rule by a formula, such as f(x) = sin(2x).

A real valued function of 2 real variables is a rule f that assigns to each ordered pair of numbers (x,y) in some set D in the plane a real number f(x,y). Usually it is given by a formula.

Example. Consider the function f(x,y) = 2x + 3xy.
Then f(3,2) would stand for the result of replacing x by 3 and y by 2. In this example, f(3,2) = 2(3) + 3(3)(2) = 24. Similarly f(2, -1) = -2.

Example. Functions of two variables can be important in biological contexts. One familiar function is the body mass index (BMI) of a person, which is used to tell whether an adult is overweight or underweight. The body mass index is defined by the formula

$$B(w,h) = 703\ w / h^2$$

where w is weight in pounds and h is height in inches. The units of B are kg/m^2.

For example, a woman who is 5 feet 7 inches and weighs 140 pounds would find her body mass index as B(140,67) = 21.9 kg/m^2 since 5 feet 7 inches means 67 inches.

The standard interpretation of the body mass index for an adult (male or female with age at least 20 years) is given in the following table:

BMI	Weight Status
Below 18.5	Underweight
18.5 – 24.9	Normal
25.0 – 29.9	Overweight
30.0 and Above	Obese

Thus the woman described in the example would be described as having normal weight. More about the BMI is available on the web at
http://www.cdc.gov/nccdphp/dnpa/bmi/index.htm

The set D is called the **domain** of the function. If we just give a formula, the **natural domain** is the set of all (x,y) for which f(x,y) makes sense.

Example. Find the natural domain of $f(x,y) = \sqrt{(4 - x^2 - y^2)}$.
Find the natural domain of f(x,y) = x / y.

In beginning calculus, the **graph** of y = f(x) is the set of all points (x,y) with y = f(x). For example, the graph of $f(x) = x^2$ is the set of all (x,y) with $y = x^2$.

If f(x,y) is a function, the **graph** of f is the picture in 3-dimensional space consisting of all (x,y,z) with z = f(x,y). Usually we can only draw an approximate picture of the graph using perspective. Typically, the graph corresponds to a surface seen in space.
Example. Sketch the graph of $z = f(x,y) = x^2+y^2$. It is called an "elliptic paraboloid".

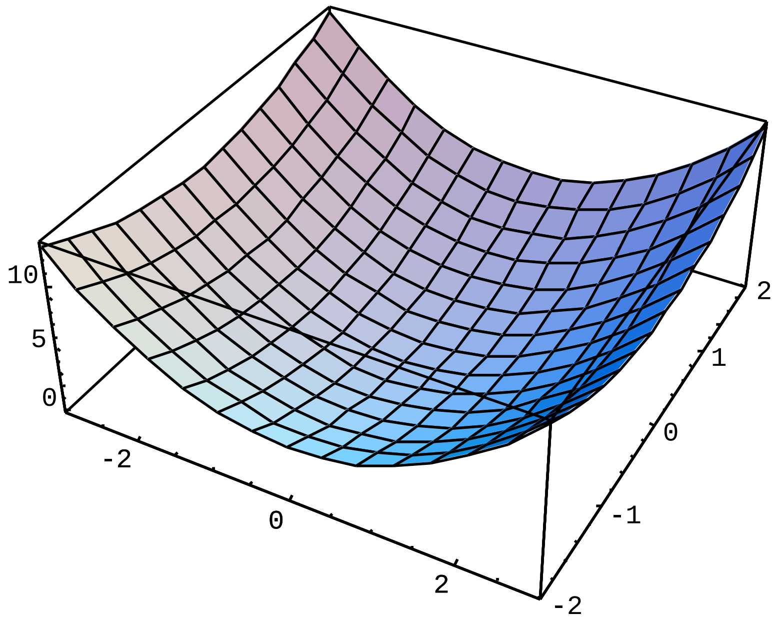

Figure 11.1.1. The graph of $z = f(x,y) = x^2+y^2$. In this graph, the x values range from -3 to 3, while the y values range from -2 to 2.

In a typical arrangement, the x-axis is represented as coming directly toward the reader out of the paper; the y-axis is represented as going to the right on the paper, and the z-axis is represented as going up on the paper.

Because of difficulties in drawing in 3 dimensions it is useful to use a different approach. If z = f(x,y) is a function, the **level curve for c** is the set of points (x,y) in the plane such that f(x,y) = c. A **contour plot** or a **contour map** is a collection of level curves with different values of c.

Example. If f(x,y) = x-y, draw a contour map.
The answer is shown in Figure 11.1.2. To find the level curve for c, we solve

$$f(x,y) = c$$
$$x-y = c$$
$$y = x - c$$

We recognize that this is the equation of the straight line with slope 1 and y-intercept - c. Thus the contour map consists of lots of lines with slope 1 and various y-intercepts.

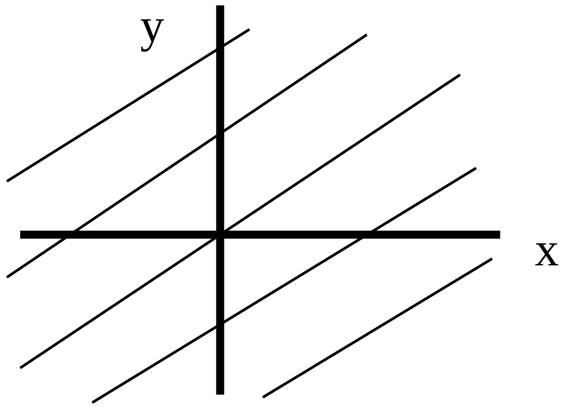

Figure 11.1.2. A contour map for f(x,y)= x - y.

Example. If $f(x,y) = x^2+y^2$, draw a contour map.
The answer is shown in Figure 11.1.3. To find the level curve for 4, we solve

$$f(x,y) = 4$$
$$x^2+y^2 = 4$$

We recognize that this is the equation of the circle centered at the origin with radius 2, since the circle of radius r and center (x0, y0) is given by the equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Similarly the level curve for 9 is

$$x^2+y^2 = 9$$

We recognize that this is the equation of the circle centered at the origin with radius 3.

Thus the contour plot consists of a collection of circles centered at the origin with various radii.

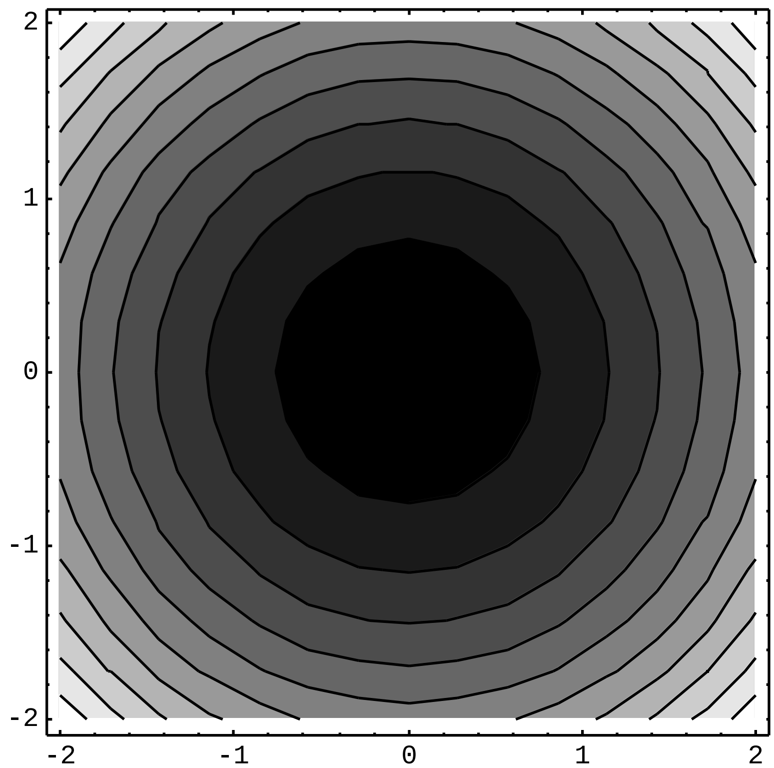

Figure 11.1.3. Contour plot for $f(x,y) = x^2+y^2$.

Example. Sketch a contour map for $g(x,y) = x^2 - y^2$.
The answer is shown in Figure 11.1.4. To find the level curve for 4, we solve

$$g(x,y) = 4$$
$$x^2-y^2 = 4$$

We recognize that this is the equation of an hyperbola. In general the level curve for c is

$$g(x,y) = c$$
$$x^2-y^2 = c$$

which is an hyperbola.

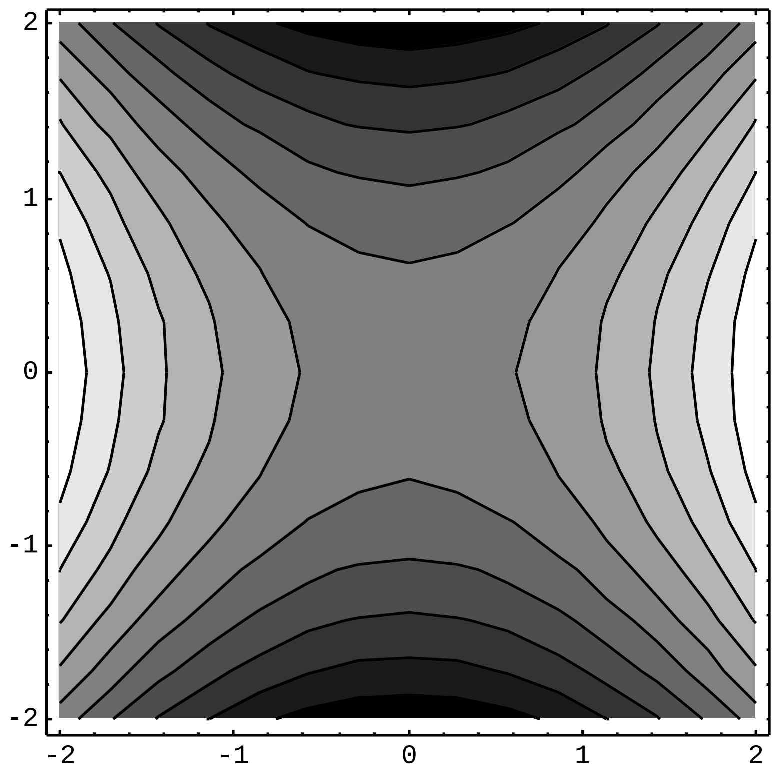

Figure 11.1.4. A contour plot for f(x,y) = x^2-y^2.

Contour plots are common in weather and engineering and maps Many newspapers and weather web pages contain a weather map with contour lines for pressure (in millibars). The curves are where the pressure is a constant c.

Functions with more than two variables are also common.

Example. The temperature T at (x,y,z) is
T = f(x,y,z) = $2x^2 + y^2 + z^2$ degrees C.
Find the temperature at the point (1, 2, 3).
Solution. The temperature is T(1,2,3) = $2(1)^2 + (2)^2 + (3)^2$ = 15 degrees C.

Example. In some complicated models, there may be functions of many variables that are needed. An example concerns blood flow in a blood vessel. When a blood vessel is narrow, the speed of the blood is slowed. The speed moreover depends on where in the blood vessel the speed is being measured. Poiseuille's Law states that the rate of flow V of blood in a blood vessel is

$$V(L, P, R, \eta) = \pi R^4 P / (8 L \eta)]$$

where V is rate of blood flow in cubic centimeters per second (cc/ sec),
R is the radius of the blood vessel in cm,
P is the pressure change in the blood vessel measured in Pascals,
η is the viscosity of the blood in Pascal seconds,
L is the length of the blood vessel in cm.

Normally blood pressure is given in mmHG =millimeters of mercury. To convert to Pascals, one multiplies the mmHg value by 133.3. A typical value for the viscosity is 0.005 Pascal sec.

For example, in a small arteriole blood vessel that is 0.05 cm long with radius 0.015 mm, the pressure drop is 50 mmHg, and the viscosity is 0.005 Pascal sec.
Then R = 0.0015 cm
L = .05 cm
P = 133.3 (50) = 6665 Pascals
η = 0.005 Pascal sec
Hence V = V(.05, 6665, .0015, .005) = 5.3×10^{-5} cc/sec.

By contrast, in a large blood vessel that is 10 cm long with radius 0.35 cm, the pressure drop is 25 mmHg, and the viscosity if 0.005 Pascal sec.
Then R = 0.35 cm
L = 10 cm
P = 133.3 (25) = 3332 Pascals
η = 0.005 Pascal sec
Hence V = 393 cc/sec.