Continuous Time Markov Chains

*Birth and Death Processes, Transition Probability Function, Kolmogorov Equations, Limiting Probabilities, Uniformization*
## Markovian Processes

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Chapter 6
Continuous Time Markov Chain

A stochastic process \( \{X(t), t \geq 0\} \) is a continuous time Markov chain (CTMC) if for all \( s, t \geq 0 \) and nonnegative integers \( i, j, x(u), 0 \leq u < s \),

\[
P \left\{ X(s + t) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s \right\} = P \left\{ X(s + t) = j \mid X(s) = i \right\}
\]

and if this probability is independent of \( s \), then the CTMC has stationary transition probabilities:

\[
P_{ij}(t) = P \left\{ X(s + t) = j \mid X(s) = i \right\} \quad \text{for all } s
\]
Alternate Definition

Each time the process enters state $i$,

The amount of time it spends in state $i$ before making a transition to a different state is \textit{exponentially distributed} with parameter $v_i$, and

When it leaves state $i$, it next enters state $j$ with probability $P_{ij}$, where $P_{ii} = 0$ and $\sum_j P_{ij} = 1$

Let $q_{ij} = v_i P_{ij}$, then $v_i = \sum_j q_{ij}$,

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \quad \text{and} \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$$
Birth and Death Processes

If a CTMC has states \{0, 1, \ldots\} and transitions from state \( n \) may go only to either state \( n - 1 \) or state \( n + 1 \), it is called a birth and death process. The birth (death) rate in state \( n \) is \( \lambda_n \) (\( \mu_n \)), so

\[
\begin{align*}
00 & = \lambda_0 \\
n & = \lambda_n + \mu_n, \ i > 0 \\
P_{01} & = 1 \\
P_{i,i+1} & = \frac{\lambda_i}{\lambda_i + \mu_i}, \ P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \ i > 0
\end{align*}
\]
Chapman-Kolmogorov Equations

“In order to get from state $i$ at time 0 to state $j$ at time $t + s$, the process must be in some state $k$ at time $t$”

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

From these can be derived two sets of differential equations:

“Backward” $P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$

“Forward” $P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$
Limiting Probabilities

If

- All states of the CTMC communicate: For each pair \(i, j\), starting in state \(i\) there is a positive probability of ever being in state \(j\), and
- The chain is positive recurrent: starting in any state, the expected time to return to that state is finite,

then limiting probabilities exist: \(P_j = \lim_{t \to \infty} P_{ij}(t)\)

(and when the limiting probabilities exist, the chain is called *ergodic*)

Can we find them by solving something like \(\pi = \pi \mathbf{P}\) for discrete time Markov chains?
Infinitesimal Generator (Rate) Matrix

Let $R$ be a matrix with elements

$$r_{ij} = \begin{cases} 
q_{ij}, & \text{if } i \neq j \\
-v_i, & \text{if } i = j 
\end{cases}$$

(the rows of $R$ sum to 0)

Let $t \to \infty$ in the forward equations. In steady state:

$$\lim_{t \to \infty} P'_{ij}(t) = \lim_{t \to \infty} \sum_{k \neq j} q_{kj}P_{ik}(t) - v_j P_{ij}(t)$$

$$0 = \sum_{k \neq j} q_{kj}P_k - v_j P_j$$

These can be written in matrix form as $PR = 0$ along with $\sum_j P_j = 1$ and solved for the limiting probabilities.

What do you get if you do the same with the backward equations?
Balance Equations

The $\textbf{PR} = 0$ equations can also be interpreted as balancing:

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

rate at which process leaves $j = \text{rate at which process enters } j$

For a birth-death process, they are equivalent to level-crossing equations

$$\lambda_n P_n = \mu_{n+1} P_{n+1}$$

rate of crossing from $n$ to $n+1 = \text{rate of crossing from } n+1 \text{ to } n$

so

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0$$

and a steady state exists if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$
Time Reversibility

A CTMC is time-reversible if and only if $P_{i}q_{ij} = P_{j}q_{ji}$ when $i \neq j$

There are two important results:

1. An ergodic birth and death process is time reversible
2. If for some set of numbers \{P_i\},
   \[
   \sum_{i} P_{i} = 1 \text{ and } P_{i}q_{ij} = P_{j}q_{ji} \text{ when } i \neq j
   \]
   then the CTMC is time-reversible and $P_{i}$ is the limiting probability of being in state $i$.

This can be a way of finding the limiting probabilities.
Uniformization

Before, we assumed that \( P_{ii} = 0 \), i.e., when the process leaves state \( i \) it always goes to a different state. Now, let \( \nu \) be any number such that \( \nu_i \leq \nu \) for all \( i \). Assume that all transitions occur at rate \( \nu \), but that in state \( i \), only the fraction \( \nu_i/\nu \) of them are real ones that lead to a different state. The rest are fictitious transitions where the process stays in state \( i \).

Using this fictitious rate, the time the process spends in state \( i \) is exponential with rate \( \nu \). When a transition occurs, it goes to state \( j \) with probability

\[
P_{ij}^* = \begin{cases} 
1 - \frac{\nu_i}{\nu}, & j = i \\
\frac{\nu_i}{\nu} P_{ij}, & j \neq i
\end{cases}
\]
Uniformization (2)

In the uniformized process, the number of transitions up to time $t$ is a Poisson process $N(t)$ with rate $\nu$. Then we can compute the transition probabilities by conditioning on $N(t)$:

$$P_{ij}(t) = P\{X(t) = j \mid X(0) = i\}$$

$$= \sum_{n=0}^{\infty} P\{X(t) = j \mid X(0) = i, N(t) = n\} P\{N(t) = n \mid X(0) = i\}$$

$$= \sum_{n=0}^{\infty} P\{X(t) = j \mid X(0) = i, N(t) = n\} \frac{e^{-\nu t} (\nu t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} P_{ij}^* \frac{e^{-\nu t} (\nu t)^n}{n!}$$
More on the Rate Matrix

Can write the backward differential equations as \( P'(t) = RP(t) \) and their solution is \( P(t) = P(0)e^{Rt} = e^{Rt} \) since \( P(0) = I \) where
\[
e^{Rt} \equiv \sum_{n=0}^{\infty} R^n \frac{t^n}{n!}
\]

but this computation is not very efficient. We can also approximate:

\[
e^{Rt} = \lim_{n \to \infty} \left( I + R \frac{t}{n} \right)^n \quad \text{or} \quad e^{Rt} \approx \left[ \left( I - R \frac{t}{n} \right)^{-1} \right]^n \quad \text{for large} \ n
\]