Continuous Time Markov Chains

Birth and Death Processes,Transition Probability Function, Kolmogorov Equations, Limiting Probabilities, Uniformization

Markovian Processes

State Space Parameter Space (Time)	Discrete	Continuous
Discrete	Markov chains (Chapter 4)	
Continuous	Continuous time Markov chains (Chapters 5, 6)	Brownian motion process (Chapter 10)

Continuous Time Markov Chain

A stochastic process $\{X(t), t \geq 0\}$ is a continuous time Markov chain (CTMC) if for all s, $t \geq 0$ and nonnegative integers i, j, $x(u)$, $0 \leq u < s$,

$$P\left\{X(s+t)=j \middle| X(s)=i, X(u)=x(u), 0 \leq u < s\right\}$$

$$= P\left\{X(s+t)=j \middle| X(s)=i\right\}$$

and if this probability is independent of s, then the CTMC has stationary transition probabilities:

$$P_{ij}(t)=P\left\{X(s+t)=j \middle| X(s)=i\right\} \text{ for all } s$$

Alternate Definition

Each time the process enters state *i*,

The amount of time it spends in state *i* before making a transition to a different state is *exponentially distributed* with parameter v_i, and

When it leaves state *i*, it next enters state *j* with probability P_{ij}, where $P_{ii} = 0$ and $\sum_j P_{ij} = 1$

Let

$$q_{ij} = v_i P_{ij}, \text{ then } v_i = \sum_j q_{ij},$$

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \text{ and } \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$$

Birth and Death Processes

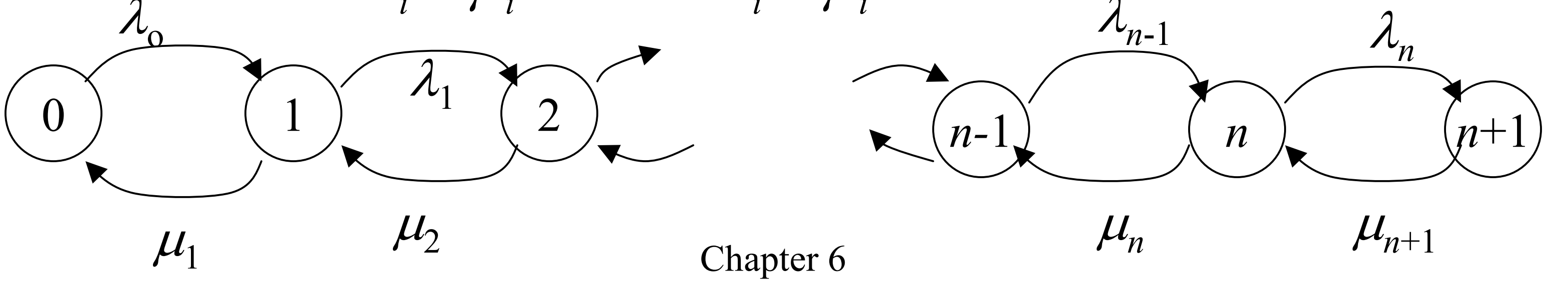
If a CTMC has states {0, 1, …} and transitions from state *n* may go only to either state *n* - 1 or state *n* + 1, it is called a birth and death process. The birth (death) rate in state *n* is λ_n (μ_n), so

$$v_0 = \lambda_0$$

$$v_i = \lambda_i + \mu_i, i > 0$$

$$P_{01} = 1$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, i > 0$$



Chapman-Kolmogorov Equations

"In order to get from state *i* at time 0 to state *j* at time *t* + *s*, the process must be in some state *k* at time *t*"

$$P_{ij}\left(t+s\right)=\sum_{k=0}^{\infty}P_{ik}\left(t\right)P_{kj}\left(s\right)$$

From these can be derived two sets of differential equations:

"Backward" $$P'_{ij}\left(t\right)=\sum_{k\neq i}q_{ik}P_{kj}\left(t\right)-v_iP_{ij}\left(t\right)$$

"Forward" $$P'_{ij}\left(t\right)=\sum_{k\neq j}q_{kj}P_{ik}\left(t\right)-v_jP_{ij}\left(t\right)$$

Limiting Probabilities

If

- All states of the CTMC communicate: For each pair *i*, *j*, starting in state *i* there is a positive probability of ever being in state *j*, and
- The chain is positive recurrent: starting in any state, the expected time to return to that state is finite,

then limiting probabilities exist: $P_j = \lim_{t \to \infty} P_{ij}\left(t\right)$

(and when the limiting probabilities exist, the chain is called *ergodic*)

Can we find them by solving something like $\pi = \pi \mathbf{P}$ for discrete time Markov chains?

Infinitesimal Generator (Rate) Matrix

Let **R** be a matrix with elements $r_{ij} = \begin{cases} q_{ij}, \text{ if } i \neq j \\ -v_i, \text{ if } i = j \end{cases}$

(the rows of **R** sum to 0)

Let $t \to \infty$ in the forward equations. In steady state:

$$\lim_{t\to\infty} P'_{ij}(t) = \lim_{t\to\infty} \sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

$$0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

These can be written in matrix form as **PR** = **0** along with $\sum_j P_j = 1$ and solved for the limiting probabilities.

What do you get if you do the same with the backward equations?

Balance Equations

The **PR = 0** equations can also be interpreted as balancing:

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k$$

rate at which process leaves $j =$ rate at which process enters j

For a birth-death process, they are equivalent to level-crossing equations $\lambda_n P_n = \mu_{n+1} P_{n+1}$

rate of crossing from n to $n+1 =$ rate of crossing from $n+1$ to n

so $P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0$ and a steady state exists if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

Time Reversibility

A CTMC is time-reversible if and only if $P_i q_{ij} = P_j q_{ji}$ when $i \neq j$

There are two important results:

1. An ergodic birth and death process is time reversible
2. If for some set of numbers $\{P_i\}$,

$$\sum_i P_i = 1 \text{ and}$$

$$P_i q_{ij} = P_j q_{ji} \text{ when } i \neq j$$

then the CTMC is time-reversible and P_i is the limiting probability of being in state *i*.

This can be a way of finding the limiting probabilities.

Uniformization

Before, we assumed that $P_{ii} = 0$, i.e., when the process leaves state i it always goes to a different state. Now, let v be any number such that $v_i \leq v$ for all i. Assume that all transitions occur at rate v, but that in state i, only the fraction v_i/v of them are real ones that lead to a different state. The rest are fictitious transitions where the process stays in state i.

Using this fictitious rate, the time the process spends in state i is exponential with rate v. When a transition occurs, it goes to state j with probability

$$P_{ij}^{*} = \begin{cases} 1 - \dfrac{v_i}{v}, \, j = i \\ \dfrac{v_i}{v} P_{ij}, \, j \neq i \end{cases}$$

Uniformization (2)

In the uniformized process, the number of transitions up to time *t* is a Poisson process *N*(*t*) with rate *v*. Then we can compute the transition probabilities by conditioning on *N*(*t*):

$$\begin{aligned}
P_{ij}(t) &= P\left\{X(t)=j \middle| X(0)=i\right\} \\
&= \sum_{n=0}^{\infty} P\left\{X(t)=j \middle| X(0)=i, N(t)=n\right\} P\left\{N(t)=n \middle| X(0)=i\right\} \\
&= \sum_{n=0}^{\infty} P\left\{X(t)=j \middle| X(0)=i, N(t)=n\right\} \frac{e^{-vt}(vt)^n}{n!} \\
&= \sum_{n=0}^{\infty} P_{ij}^{*n} \frac{e^{-vt}(vt)^n}{n!}
\end{aligned}$$

More on the Rate Matrix

Can write the backward differential equations as $\mathbf{P}'(t) = \mathbf{R}\mathbf{P}(t)$ and their solution is $\mathbf{P}(t) = \mathbf{P}(0)e^{\mathbf{R}t} = e^{\mathbf{R}t}$ since $\mathbf{P}(0) = \mathbf{I}$ where

$$e^{\mathbf{R}t} \equiv \sum_{n=0}^{\infty} \mathbf{R}^n \frac{t^n}{n!}$$

but this computation is not very efficient. We can also approximate:

$$e^{\mathbf{R}t} = \lim_{n\to\infty}\left(\mathbf{I} + \mathbf{R}\frac{t}{n}\right)^n \quad \text{or } e^{\mathbf{R}t} \approx \left[\left(\mathbf{I} - \mathbf{R}\frac{t}{n}\right)^{-1}\right]^n \text{ for large } n$$