Exponential Distribution & Poisson Process

Memorylessness & other exponential distribution properties; Poisson process; Nonhomogeneous & compound P.P.’s
Chapter 5

Exponential Distribution: Basic Facts

- **Density** $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $\lambda > 0$

- **CDF** $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

- **MGF** $\phi(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}$

- **Mean** $E[X] = \frac{1}{\lambda}$

- **Variance** $\text{Var}[X] = \frac{1}{\lambda^2}$

  Coefficient of variation

  $\frac{E[X]}{\sqrt{\text{Var}[X]}} = 1$
Key Property: **Memorylessness**

\[ P\{X > s + t \mid X > t\} = P\{X > s\} \quad \text{for all } s, t \geq 0 \]

- Reliability: Amount of time a component has been in service has no effect on the amount of time until it fails.
- Inter-event times: Amount of time since the last event contains no information about the amount of time until the next event.
- Service times: Amount of remaining service time is independent of the amount of service time elapsed so far.
Other Useful Properties

*Sum* of *n* independent exponential r.v.’s with common parameter *λ* has a gamma distribution w/parameters (*n*, *λ*)

**Competing Exponentials:**
If *X*₁ and *X*₂ are independent exponential r.v.’s with parameters *λ*₁ and *λ*₂, resp., then

\[ P\{ X_1 < X_2 \} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \]

(guesserizes to any number of competing r.v.’s)

**Minimum of exponentials:**
If *X*₁, *X*₂, ..., *X*ₙ are independent exponential r.v.’s where *X*ₙ has parameter *λ*ᵢ, then

\[ \min(X_1, X_2, ..., X_n) \text{ is exponential w/parameter } \lambda_1 + \lambda_2 + ... + \lambda_n \]
Counting Process

A stochastic process \( \{N(t), t \geq 0\} \) is a \textit{counting process} if \( N(t) \) represents the total number of events that have occurred in \([0, t]\)

Then \( \{N(t), t \geq 0\} \) must satisfy:
- \( N(t) \geq 0 \)
- \( N(t) \) is an integer for all \( t \)
- If \( s < t \), then \( N(s) \leq N(t) \)
- For \( s < t \), \( N(t) - N(s) \) is the number of events that occur in the interval \((s, t]\).
Stationary & Independent Increments

- A counting process has *independent increments* if, for any $0 \leq s < t \leq u < v$, $N(t) - N(s)$ is independent of $N(v) - N(u)$. That is, the numbers of events that occur in nonoverlapping intervals are independent random variables.

- A counting process has *stationary increments* if the distribution if, for any $s < t$, the distribution of $N(t) - N(s)$ depends only on the length of the time interval, $t - s$. 
Poisson Process Definition 1

A counting process \( \{N(t), t \geq 0\} \) is a *Poisson process with rate \( \lambda, \lambda > 0 \), if*

\[ N(0) = 0 \]

The process has independent increments

The number of events in any interval of length \( t \) follows a Poisson distribution with mean \( \lambda t \) (therefore, it has stationary increments), i.e.,

\[
P\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \ n = 0,1,\ldots
\]
Poisson Process Definition 2

A function \( f \) is said to be \( o(h) \) ("Little oh of h") if 
\[
\lim_{h \to 0} \frac{f(h)}{h} = 0
\]

A counting process \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda, \lambda > 0 \), if

\( N(0) = 0 \)

The process has stationary and independent increments

\[
P\{N(h) = 1\} = \lambda h + o(h)
\]

\[
P\{N(h) \geq 2\} = o(h)
\]

Definitions 1 and 2 are equivalent!
Interarrival and Waiting Times

The times between arrivals $T_1, T_2, \ldots$ are independent exponential r.v.’s with mean $1/\lambda$:

- $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$
- $P\{T_2 > t | T_1 = s\} = e^{-\lambda t}$

The (total) waiting time until the $n$th event has a gamma dist’n: $S_n = \sum_{i=1}^{n} T_i$
Other Poisson Process Properties

Poisson Splitting:
Suppose \( \{N(t), t \geq 0\} \) is a P.P. with rate \( \lambda \), and suppose that each time an event occurs, it is classified as type I with probability \( p \) and type II with probability \( 1-p \), independently of all other events. Let \( N_1(t) \) and \( N_2(t) \), respectively, be the number of type I and type II events up to time \( t \).

Then \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) are independent Poisson processes with respective rates \( \lambda p \) and \( \lambda(1-p) \).
Other Poisson Process Properties

Competing Poisson Processes:

Suppose \{N_1(t), t \geq 0\} and \{N_2(t), t \geq 0\} are independent Poisson processes with respective rates \( \lambda_1 \) and \( \lambda_2 \).

Let \( S_n^i \) be the time of the \( n \)th event of process \( i \), \( i = 1, 2 \).

\[
P\{S_n^1 < S_m^2\} = \sum_{k=n}^{n+m-1} \binom{n + m - 1}{k} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^{n+m-1-k}
\]
Other Poisson Process Properties

If $Y_1, Y_2, \ldots, Y_n$ are random variables, then $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are their order statistics if $Y_{(k)}$ is the $k$th smallest value among $Y_1, Y_2, \ldots, Y_n,$ $k = 1, \ldots, n.$

**Conditional Distribution of Arrival Times:**

Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda$ and for some time $t$ we know that $N(t) = n$. Then the arrival times $S_1, S_2, \ldots, S_n$ have the same conditional distribution as the order statistics of $n$ independent uniform random variables on $(0, t)$. 
Nonhomogeneous Poisson Process

A counting process \( \{N(t), t \geq 0\} \) is a nonhomogeneous Poisson process with intensity function \( \lambda(t), t \geq 0 \), if:

\[ N(0) = 0 \]

The process has independent increments (not stationary incr.)

\[ P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h) \]

\[ P\{N(t+h) - N(t) \geq 2\} = o(h) \]

Let \( m(t) = \int_0^t \lambda(y)dy \)

Then

\[ P\{N(t+s) - N(s) = n\} = e^{-[m(s+t)-m(s)]} \frac{(m(s+t)-m(s))^n}{n!}, n = 0,1,... \]
Compound Poisson Process

A counting process \( \{X(t), \ t \geq 0\} \) is a compound Poisson process if:

\[
X(t) = \sum_{i=1}^{N(t)} Y_i, \ t \geq 0
\]

where \( \{N(t), \ t \geq 0\} \) is a Poisson process and \( \{Y_i, \ i = 1, 2, \ldots\} \) are independent, identically distributed r.v.’s that are independent of \( \{N(t), \ t \geq 0\} \).

By conditioning on \( N(t) \), we can obtain:

\[
E\left[ X(t) \right] = \lambda t E [Y_1]
\]

\[
\text{Var}\left[ X(t) \right] = \lambda t E \left[ Y_1^2 \right]
\]