Exponential Distribution & Poisson Process

Memorylessness & other exponential distribution properties; Poisson process; Nonhomogeneous & compound P.P.'s

Exponential Distribution: Basic Facts

- Density $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \lambda > 0$
- CDF $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$
- MGF $\phi(t) = E\left[e^{tX}\right] = \dfrac{\lambda}{\lambda - t}$
- Mean $E[X] = \dfrac{1}{\lambda}$
- Variance $\text{Var}[X] = \dfrac{1}{\lambda^2}$

Coefficient of variation

$$\frac{E[X]}{\sqrt{\text{Var}[X]}} = 1$$

Key Property: *Memorylessness*

$$P\left\{X > s+t \middle| X > t\right\} = P\left\{X > s\right\} \quad \text{for all } s, t \geq 0$$

- Reliability: Amount of time a component has been in service has no effect on the amount of time until it fails
- Inter-event times: Amount of time since the last event contains no information about the amount of time until the next event
- Service times: Amount of remaining service time is independent of the amount of service time elapsed so far

Other Useful Properties

Sum of n independent exponential r.v.'s with common parameter λ has a gamma distribution w/parameters (n, λ)

Competing Exponentials:

If X_1 and X_2 are independent exponential r.v.'s with parameters λ_1 and λ_2, resp., then

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

(generalizes to any number of competing r.v.'s)

Minimum of exponentials:

If $X_1, X_2, \ldots, X_n$ are independent exponential r.v.'s where X_n has parameter λ_i, then

min($X_1, X_2, \ldots, X_n$) is exponential w/parameter

$\lambda_1 + \lambda_2 + \ldots + \lambda_n$

Counting Process

A stochastic process $\{N(t), t \geq 0\}$ is a *counting process* if $N(t)$ represents the total number of events that have occurred in $[0, t]$

Then $\{N(t), t \geq 0\}$ must satisfy:

$N(t) \geq 0$

$N(t)$ is an integer for all t

If $s < t$, then $N(s) \leq N(t)$

For $s < t$, $N(t) - N(s)$ is the number of events that occur in the interval $(s, t]$.

Stationary & Independent Increments

- A counting process has *independent increments* if, for any $0 \le s < t \le u < v$, $N(t) - N(s)$ is independent of $N(v) - N(u)$. That is, the numbers of events that occur in nonoverlapping intervals are independent random variables.

- A counting process has *stationary increments* if the distribution if, for any $s < t$, the distribution of $N(t) - N(s)$ depends only on the length of the time interval, $t - s$.

Poisson Process Definition 1

A counting process $\{N(t), t \geq 0\}$ is a *Poisson process with rate* λ, $\lambda > 0$, if

$N(0) = 0$

The process has independent increments

The number of events in any interval of length t follows a Poisson distribution with mean λt (therefore, it has stationary increments), i.e.,

$$P\left\{N\left(t+s\right)-N\left(s\right)=n\right\}=\frac{e^{-\lambda t}\left(\lambda t\right)^{n}}{n!}, n=0,1,\ldots$$

Poisson Process Definition 2

A function f is said to be $o(h)$ ("Little oh of h") if $\lim_{h \to 0} \frac{f(h)}{h} = 0$

A counting process $\{N(t), t \geq 0\}$ is a *Poisson process with rate* λ, $\lambda > 0$, if

$N(0) = 0$

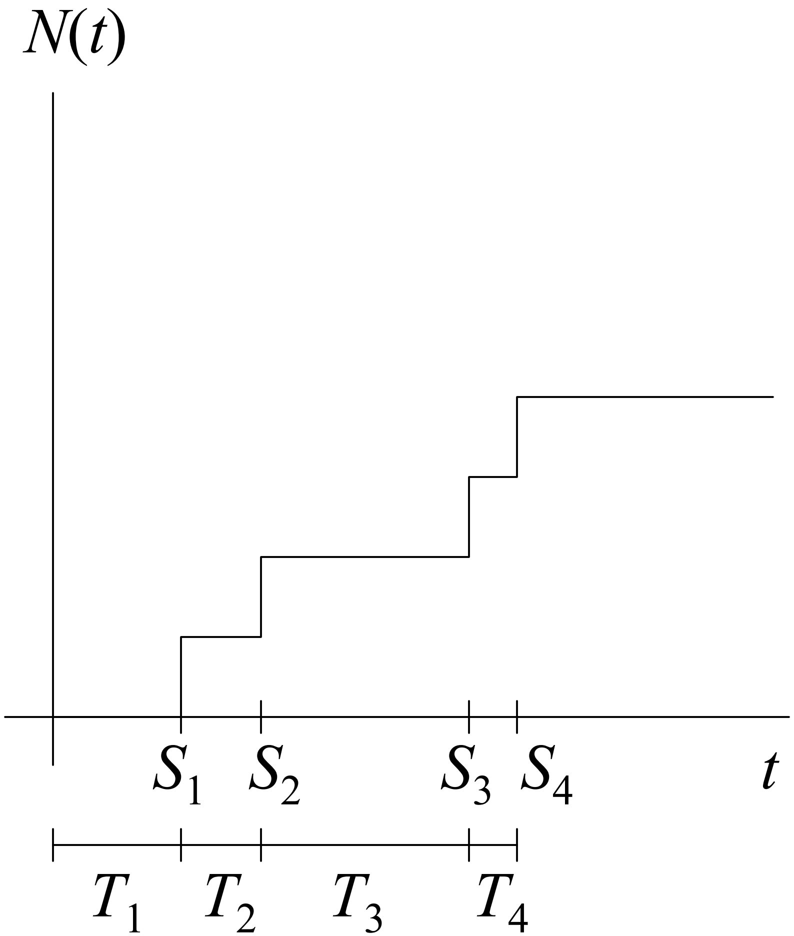
The process has stationary and independent increments

$$P\{N(h) = 1\} = \lambda h + o(h)$$

$$P\{N(h) \geq 2\} = o(h)$$

Definitions 1 and 2 are equivalent!

Interarrival and Waiting Times



The times between arrivals T_1, T_2, … are independent exponential r.v.'s with mean $1/\lambda$:

$$P\left\{T_1 > t\right\} = P\left\{N\left(t\right) = 0\right\} = e^{-\lambda t}$$

$$P\left\{T_2 > t \middle| T_1 = s\right\} = e^{-\lambda t}$$

The (total) waiting time until the nth event has a gamma dist'n: $S_n = \sum_{i=1}^{n} T_i$

Other Poisson Process Properties

Poisson Splitting:

Suppose $\{N(t), t \geq 0\}$ is a P.P. with rate λ, and suppose that each time an event occurs, it is classified as type I with probability p and type II with probability 1-p, independently of all other events. Let $N_1(t)$ and $N_2(t)$, respectively, be the number of type I and type II events up to time t.

Then $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are *independent* Poisson processes with respective rates λp and $\lambda(1-p)$.

Other Poisson Process Properties

Competing Poisson Processes:

Suppose $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with respective rates λ_1 and λ_2.

Let S_n^i be the time of the nth event of process i, $i = 1,2$.

$$P\left\{S_n^1 < S_m^2\right\} = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$

Other Poisson Process Properties

If $Y_1, Y_2, \ldots, Y_n$ are random variables, then $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are their order statistics if $Y_{(k)}$ is the kth smallest value among $Y_1, Y_2, \ldots, Y_n$, $k = 1, \ldots, n$.

Conditional Distribution of Arrival Times:

Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate λ and for some time t we know that $N(t) = n$. Then the arrival times $S_1, S_2, \ldots, S_n$ have the same conditional distribution as the order statistics of n independent uniform random variables on $(0, t)$.

Nonhomogeneous Poisson Process

A counting process $\{N(t),\ t \geq 0\}$ is a *nonhomogeneous Poisson process with intensity function* $\lambda(t),\ t \geq 0$, if:

$N(0) = 0$

The process has independent increments (*not* stationary incr.)

$$P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$$

$$P\{N(t+h) - N(t) \geq 2\} = o(h)$$

Let $m(t) = \int_0^t \lambda(y)dy$

Then

$$P\{N(t+s) - N(s) = n\} = e^{-[m(s+t)-m(s)]}\frac{(m(s+t)-m(s))^n}{n!},\ n = 0,1,\ldots$$

Compound Poisson Process

A counting process $\{X(t), t \geq 0\}$ is a *compound Poisson process* if:

$$X(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0$$

where $\{N(t), t \geq 0\}$ is a Poisson process and $\{Y_i, i = 1, 2, \ldots\}$ are independent, identically distributed r.v.'s that are independent of $\{N(t), t \geq 0\}$.

By conditioning on $N(t)$, we can obtain:

$$E\left[X(t)\right] = \lambda t E\left[Y_1\right]$$

$$\text{Var}\left[X(t)\right] = \lambda t E\left[Y_1^2\right]$$