A Markov Decision Model to Evaluate Outsourcing in Reverse Logistics

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Abstract.

One of the most important decisions regarding reverse logistics (RL) is whether to outsource such functions or not, due to the fact that RL does not represent a production or distribution firm’s core activity. To explore the hypothesis that outsourcing RL functions is more suitable when returns are more variable, we formulate and analyze a Markov decision model of the outsourcing decision. The reward function includes capacity and operating costs of either performing RL functions internally or outsourcing them, and the transitions among states reflect both the sequence of decisions taken and a simple characterization of the random pattern of returns over time. We identify nonrestrictive conditions on the cost parameters and the return rate that guarantee the existence of an optimal threshold policy for outsourcing. Under mild assumptions, this threshold is more likely to be crossed, the higher the uncertainty in returns. A set of numerical examples illustrate how the threshold for outsourcing decreases while the probability of crossing any fixed threshold increases with the variability in the return volume. They also indicate that the threshold is more easily crossed when the length of the product’s life cycle is shorter.

Keywords
Reverse Logistics, Outsourcing, Markov Decision Model, Monotone Policy, Product Life Cycle.
\textbf{Introduction.}

Once a firm distributes its products to retailers and final consumers, its flow of materials has not stopped: a substantial return flow of products may occur due to either generous return policies or legislation that requires producers to accept responsibility for their products at end-of-life. We consider reverse logistics (RL) to include all activities associated with collecting, inspecting, reprocessing, redistributing, and disposing of items after they were originally sold (see Figure 1). Although it has long been perceived as a nuisance, it has recently been considered as an improvement area with the correct focus. Every manufacturing, distribution or sales firm, irrespective of its size, types of products or geographic location, can benefit from planning, implementing and controlling RL activities. Unfortunately, not enough analytical models that assist in RL strategic decisions currently exist.

Given that RL is not the firm’s core activity, one of the most important decisions to be taken by any producer is whether or not to outsource such functions. Any organization might decide whether to perform the RL functions internally, or to contract with a third-party reverse logistics provider (3PRLP) to perform them. This decision is mostly identified as a “take it or leave it” alternative, because the chosen strategy, once adopted, will not be changed frequently. The management of returns is complicated by the substantial uncertainties associated with their timing, volume and condition. This paper focuses on how the uncertainty in returns affects the decision of whether or not to outsource their RL management.

Our central hypothesis is that outsourcing RL is more suitable when returns are more variable. This hypothesis arose from a qualitative analysis of the published literature on outsourcing and RL, which is overviewed briefly in the next section. In Section 3, we formulate and analyze a Markov decision model of the outsourcing decision. The reward
function includes the most significant components of the cost of either performing RL functions internally or outsourcing them, and the transitions among states reflect both the sequence of decisions taken and a simple characterization of the random pattern of returns over time. We assume that RL functions initially are performed internally. In order to focus simply on the outsourcing decision, we limit our attention to two possible actions in each period: either adjust internal capacity to match the expected number of returns in the next period, or switch permanently to outsourcing. By analyzing the cost and transition probability functions, we identify nonrestrictive conditions for the existence of an optimal monotone policy over the partially ordered state space, which reduces to a threshold of cumulative returns, beyond which outsourcing is optimal. Such conditions are stated in terms of the cost parameters involved, as well as the return rate for the product considered. Finally, we show that under mild assumptions, this threshold is more likely to be crossed when the uncertainty in returns is higher. Numerical examples that illustrate how the threshold for outsourcing decreases while the probability of crossing any fixed threshold increases with the variability in the return volume lend further support to the hypothesis. These numerical examples demonstrate not only how the threshold is easily crossed when the variability on the return volume increases, but also when the length of the product’s life cycle is shorter.

Finally, in Section 4, we draw conclusions and outline future research that can be developed based on this work.

2. Literature Review on Outsourcing RL Functions

There are many reasons why products are returned, either by consumers or by the companies involved in the distribution chain. Retailers may return products because of damage in transit, expired date code, the model being discontinued or replaced, seasonality, excessive retailer inventories, retailer going out of business, etc. On the other hand, consumers can return
products for such reasons as quality problems, failure to meet the consumer’s needs, for remanufacturing, or for proper disposal.

Also, once products have reached the end of their useful life, they may be able to be remanufactured, refurbished or repaired; thus extending their life. These options can provide significant benefits in some instances, especially for products that have modular components (e.g. electronic equipment, computers) that can be replaced, upgraded and/or refurbished. The value of items that are remanufactured typically will be lower than that of the same items produced for the first time. However, their value will be substantially higher than that of items being sold for scrap, salvage or recycling (Stock, 1998).

The importance of RL has increased in recent years. Currently, estimates of annual sales of remanufactured products exceed $50 billion in the United States alone (Guide and van Wassenhove, 2003). There are no worldwide estimates of the economic scope of reuse activities, but the number of firms engaged in this sector is growing rapidly in response to the opportunities to create additional wealth, and in response to the enactment of extended producer responsibility legislation in several countries. Unfortunately, even with this significant development of the RL market in recent years, not enough analytical models that assist in RL strategic decisions currently exist. In a survey of current literature, Dowlatshahi (2005) identified the present state of theory in RL.

A number of researchers have addressed problems and opportunities in RL management. Guide and van Wassenhove (2003) argued that closed-loop supply chains (which are composed of the typical forward-supply network and the RL network) can be viewed as a business proposition where profit maximization is the objective. The characteristics of such maximization will depend on the forward supply chain characteristics, as well as the RL network composition. Some recent research considers both forward and reverse activities simultaneously, e.g., Vaidyanathan (2006) adopted an analytical approach
to address production planning in closed-loop supply chains, with product recovery and reuse. However, the management of RL activities is complicated by factors that are less prevalent in the forward supply chain.

Uncertainty in product returns complicates several aspects of RL management. For example, recent work by Nakashima et al. (2004) illustrated how this uncertainty, which they characterized in terms of a virtual inventory level, can be modeled in a Markov decision process for controlling a remanufacturing system.

The product life cycle is another critical factor regarding RL systems. Tibben-Lembke (2002) clearly explained its importance in the analysis of RL systems. In this vein, Bufardi et al. (2004) developed a multicriteria decision-aid approach for product end-of-life alternative selection. Also, Gonzalez et al. (2005) developed a new approach for enhancing reuse alternatives to reduce environmental problems from the earliest stages of product life cycle.

Frequently, uncertainty in returns combines with a short product life cycle to increase the complexity of decisions that may have significant economic impact. Tang et al. (2004) performed a detailed economic evaluation of disassembly processes for remanufacturing systems. Also, Willems et al. (2006) developed a linear programming approach to quantifying the turning point to make disassembly economically viable. The cost of managing a returned item is one of the most important factors when choosing how to dispose of it, as well as the price to be received for it, if such a price exists. The factors to consider will differ according to the characteristics of the RL system, such as its size, characteristics, products manufactured, managerial strategies and goals, etc. The amount of money invested in these activities will be a critical issue too.

Outsourcing to a 3PRLP has been identified as one of the most important management strategies for RL networks in the recent years. Razzaque and Sheng (1998) surveyed the literature related to outsourcing logistics functions. With regard to RL functions, Meade and
Sarkis (2002) stated three different choices that can be made: to do nothing, to develop an internal RL function, or to find a 3PRLP and partner with them. They developed a model for selecting and evaluating 3PRLP. However, this model does not represent a tool for determining whether or not to outsource RL activities, but rather it helps in the decision of selecting a 3PRLP once the firm has chosen the outsourcing strategy. Krumwiede and Sheu (2002) showed a particular model for market entry by a 3PRLP, which helps those companies who would like to pursue RL as a new market. However, Dowlatshahi (2000) warned that some such firms are not really prepared to effectively address these service needs due to the lack of knowledge of RL networks.

One of the most important issues is to define whether the firm considers RL activities as part of its core functions. When this is not the case, outsourcing might represent a good alternative in order to allow the firm to focus on its core activities (Wu, et al., 2005).

In a detailed qualitative analysis, Serrato (2006) found that some of the most important 3PRLPs are located in industry sectors with high return variability, as well as a short length for the product life cycle. Due to the extremely high variability in the rate of returns for the products managed in these RL systems, it is not economically feasible for a firm to maintain its own RL facilities to deal with that flow, given that the amount of units to be returned will be significantly uncertain over time, and the required capacity will be changing constantly. The complexity of this situation increases when the life cycle for this type of products is extremely short, which requires quick but adequate decisions for these RL systems, in order to efficiently respond to such changing conditions. This response can be accomplished effectively by involving a 3PRLP, which specializes in these activities, and can take advantage of the economies of scale to convert RL functions into a profit-creating activity in the closed-loop chain.
On the other hand, not many 3PRLPs are active in industry sectors with lower return variability and longer product life cycles, because it is easier for the producer to develop its own facilities to deal with the return flow, even though RL may not be part of its core activities. The relatively low uncertainty in the amount of returns, and the longer time periods for planning, developing and implementing RL systems, allow these firms to implement their own RL systems without a particular need for another party involved. Serrato (2006) developed a detailed analysis of these conclusions regarding outsourcing RL functions.

3. Markov Decision Model.

The MDM is designed to include the major cost drivers in the outsourcing decision, the uncertainty in the return volume, temporal variability in sales, and the impracticality of multiple transitions between performing RL functions internally and outsourcing them. To simplify the focus on return volume variability, we assume that sales can be estimated accurately from historical data related to this scenario and therefore are known. Returns depend on the amount of units previously sold and the fraction of them that will be returned through the firm’s RL system. Define the following notation:

- $L =$ Length of the product life cycle, which will depends on the particular RL scenario considered.
- $W =$ Time length defined by the firm to continue managing the returns for the product analyzed, after the last sale was made.
- $T =$ Length of the horizon analysis, $T=L+W$.
- $t =$ Decision epoch, $t = 1, \ldots, T − 1$, where decision epoch $t$ represents the end of period $t$.

Time $T$ corresponds to the end of the problem horizon, where no decision is taken.

- $s_t =$ Amount of units sold by the firm during period $t$. 
$S_t = \text{Cumulative sales experienced by the firm from period 1 through the end of period } t,$

\[ S_t = \sum_{i=1}^{t} s_i. \]

$r = \text{Return rate, i.e., the expected fraction of units previously sold but not yet returned that will be returned in the next period.}$

$x_t = \text{Amount of units returned in period } t.$

$w_t = \text{Cumulative amount of units returned from period 1 to the end of period } t, \quad w_t = \sum_{i=1}^{t} x_i.$

$k_t = \text{RL capacity held by the firm at the beginning of period } t, \text{ which represents the number of units that can be processed in a single period.}$

$n_t = \text{Number of units outstanding in the market at the end of period } t, \quad n_t = S_t - w_t.$

The following assumptions underlie the MDM:

Assumption 1: The sales in each period of the study horizon are known.

Assumption 2: Each item that has been sold but not returned has a fixed probability, \(r\), of being returned in the next period, independent of all other items. This is consistent with Toktay et al. (2003), where the number of periods between when a product was sold and when it was returned was modeled as a geometrically distributed random variable. It follows that knowing \(n_t\) at time \(t\), the number of returns in period \(t+1\) has a binomial distribution with parameters \(n_t\) and \(r\), such that the expected amount of returns in the next period can be obtained as:

\[ E(x_{t+1}) = n_t r, \quad (1) \]

and the variability in the number of returns is:

\[ Var(x_{t+1}) = n_t r (1 - r). \quad (2) \]

Note that the variability increases as \(n_t\) increases, for fixed \(r\). Then, the variability in the return volume increases as the number of outstanding units increases. The variability also
increases as $r$ approaches 0.5 from below. However, as Rogers and Tibben-Lembke (1999) observe, the return rate in most industries is between zero and 0.3, approaching 0.5 only in some specific industry sectors.

Assumption 3: The firm’s RL capacity is continuous; i.e., it can be added or subtracted in any quantity. This implies that the policy followed by any firm when adjusting its RL capacity can consider any number of capacity units. However, to simplify the model and focus on the structure of an optimal policy, we assume that if reverse logistics functions are carried out internally, then the capacity will be adjusted to equal the expected number of returns in the next period.

Assumption 4: If the number of returns in a period exceeds the RL capacity, the firm pays a shortage penalty, which represents the cost of either disposal or outsourcing the return processing on a temporary, emergency basis. No returns are carried over to a future period to be processed later. This assumption is relevant when returns are economically perishable, so that the positive value to be gained from handling them promptly is lost or greatly diminished by delay; or when storage is not physically feasible.

Additional assumptions concerning costs are given later in this section.

3.1. Model Definition.
3.1.1. States.

The system state at each decision epoch $t$ is defined as:

$$(k_t, w_t) \text{ for } t = 1, 2, ..., T$$

where $k_t$ represents the RL capacity owned by the firm during period $t$, measured in units per period and $w_t$ is the cumulative number of returns through the end of period $t$. As described below, the system states are partially ordered according to $w_t$. At decision epoch 0, the system
state is \((k_0, w_0 = 0)\); i.e., given an initial RL capacity \(k_0\), no returns are yet experienced by the firm.

3.1.2. Actions.

Given that the purpose of the MDM is to determine whether and when to outsource, it is assumed that at the end of any period \(t\), either of the following actions can be taken:

\(a = 0\): Continue performing the RL activities internally by updating the firm’s capacity to the expected amount of returns in the next period:

\[
k_{t+1} = E(x_{t+1}) = n_r r
\]

\(a = 1\): Adopt an outsourcing strategy for the RL activities by having a 3PRLP perform such activities and taking the firm’s RL capacity to zero; i.e., \(k_{t+1} = 0\). Given that RL does not represent a core activity for the firm, it is also assumed that once the outsourcing decision is taken, it remains in place for the rest of the problem horizon.

Given that \(n_t\) is an integer and that the capacity levels \(k_{t+1}\) are adjusted according to equation (3) over a finite horizon, the problem has a discrete state space.

3.1.3. Transition Probabilities.

As the returns in each period follow a binomial distribution derived from the system state, and given that the sales function is also known, the transition probability values between states are defined as:

\[
p_{t+1} \left[ (k_{t+1}, w_{t+1}) \mid (k_t, w_t), a \right] \quad \text{for} \quad a \in \{0, 1\},
\]

where for \(a = 0\) we have:

\[
p_{t+1} \left[ (n_r, w_r + j) \mid (k_t, w_t), 0 \right] = \begin{cases} \binom{n_t}{j} r^j (1-r)^{n_t-j} & \text{for } j = 0, 1, \ldots, n_t \\ 0 & \text{otherwise} \end{cases}
\]
and for $a = 1$ we have:

$$
p_{i,j}[(0, w_i + j)| (k_i, w_i), 1] = \begin{cases} 
\binom{n_i}{j} r^j (1 - r)^{n_i-j} & \text{for } j = 0,1,\ldots,n_i \\
0 & \text{otherwise.}
\end{cases}
$$

That is, the action taken determines the next period’s capacity, but the second state variable $w_{i+1}$ depends only on $n_i = S_i - w_i$ according to the binomial distribution for the returns.

3.1.5. Rewards.

Define the following set of costs, where a capacity unit represents firm’s ability to process one returned item during a single period:

- $c_1$: Unit investment cost for increasing the firm’s capacity ($/capacity unit).
- $c_2$: Unit capacity disinvestment cost ($/capacity unit).
- $c_3$: Fixed internal cost ($/capacity unit/period).
- $c_4$: Unit internal labor cost ($/unit).
- $c_5$: Unit shortage cost ($/unit).
- $c_6$: Unit capacity salvage value ($/capacity unit).
- $c_7$: Unit outsourcing cost ($/unit).

We assume $c_1, c_3, c_4, c_5, c_7 > 0$ because they represent costs for the firm, while $c_2$ and $c_6$ are unrestricted in sign, which allows the net cost of contracting capacity and salvaging equipment to be positive or negative. Figure 1 shows where these costs are located in the RL chain.

Figure 1. Relationship between RL chain and costs considered in the MDM.

Given that RL does not represent a core activity for the firm, profits from remanufacturing are not considered. The next relationships are assumed between these cost parameters:
First, (7) implies that what is obtained when capacity is contracted is less than what was invested to expand it; i.e., there can be no profit from simply expanding and later contracting capacity. Inequality (8) states that the cost of decreasing the firm’s capacity no greater than the cost of maintaining it for an additional period.

Also, (9) is reasonable because \( c_7 \) must cover both fixed and variable costs for the 3PRLP, where \( c_4 \) consists only of the variable cost for the firm. But if economies of scale are considered (as should be, given that RL is a core activity for the 3PRLP), fixed costs per unit for the 3PRLP are lower than fixed costs per unit for the firm. Also, (10) is relevant because otherwise, all the 3PRLP’s potential clients could keep their own capacity low and just pay the shortage cost rather than following an outsourcing option. On the other hand, inequality (11) represents a motivation to develop internal capacity, given that the total internal cost of maintaining the capacity for one additional period and then processing one additional unit is less than the shortage cost for that unit.

With these cost parameters, the following cost structure is defined for actions \( a = 0 \) or 1. For \( a = 0 \), we have:

\[
R_{i+1}((k_i,w_i),0) = -c_1(n_r-k_i) - c_2(k_i-n_r) - c_3n_r - c_4E\left[\min(x_{i+1},n_r)\right] - c_5E\left[(x_{i+1}-n_r)^+\right],
\]

where \((\cdot)^+\) denotes \(\max(\cdot,0)\) and it is assumed that any unit that was not managed through the RL system in the period it was taken back, is lost and will not be remanufactured later. For \( a = 1 \),
\[
R_{t+1}((k_t, w_t), 1) = c_0 k_t - c_7 \left( n_t + \sum_{i=t+1}^{T} s_i \right), \quad \text{where} \sum_{i=t+1}^{T} s_i = 0 \quad \text{if} \quad t + 1 > L
\]

Here, \( c_7 \) corresponds to the payment made to the 3PRLP for the expected returns from period \( t+1 \) onwards. Recall that, given that RL is not a core activity, it is assumed that when taken, the outsourcing option will remain in effect for the remainder of the planning horizon. Recall also from assumption 1, that the future sales can also be estimated accurately. This function also implies that the 3PRLP has infinite capacity, given the fact that RL does represent a core activity for it.

Also, we have the terminal reward in period \( T \):

\[
R_T (k_T, w_T, a) = c_0 k_T - c_3 n_T, \quad \text{for} \quad a \in \{0, 1\} \quad \text{and} \quad k_T > 0
\]

because the RL capacity defined by the firm is taken to zero in the last period, incurring the corresponding salvage value. Also, this function reflects the cost incurred by not being able to remanufacture any expected returned unit during period \( T \) or later.

### 3.2. System dynamics.

During each period \( t \) the system:

1. Has capacity in the amount of \( k_{t-1} \) at the beginning of the period, \( w_{t-1} \) units have been returned, and there are \( n_{t-1} \) units that are still in the market (were already sold and have not been returned);
2. Computes the expectation \( E(x_t)=n_{t-1}r \) for the returns and applies a control \( \delta_t(k_{t-1}, w_{t-1})=0 \) or 1;
3. If \( \delta_t(k_{t-1}, w_{t-1}) = 0 \), \( k_t \) is set equal to \( E(x_t) \) and the firm incurs either an investment cost \( c_1 (n_{t-1}r - k_{t-1})^+ \), or a disinvestment cost \( c_2 (k_{t-1} - n_{t-1}r)^+ \) by adjusting the capacity, as well as a fixed cost \( c_3 n_{t-1}r \);
4. If \( \delta_t(k_{t-1}, w_{t-1}) = 1 \), \( k_t \) is set equal to zero and the firm incurs a salvage value \( c_6 k_{t-1} \);
5. Experiences a random amount of returns $x_i$, which determines the new system state 
\[(k_i, w_i = w_{i-1} + x_i),\] as well as an amount of sales $s_i$, which determines the new 
cumulative sales level for the firm \((S_t = S_{t-1} + s_t)\).

6. Incurs either an internal or a shortage cost, 
\[c_\text{int} \min (x_i, n_{i-1} r_i), \quad \text{or} \quad c_\text{short} (x_i - n_{i-1} r_i)^+,\]
respectively, if \(\delta(k_{i-1}, w_{i-1}) = 1\) and outsourcing cost 
\[c_\gamma \left( n_{i-1} + \sum_{t=0}^{T} s_t \right),\] otherwise.

Given an initial system state \((k_0, w_0) = 0\), the problem is to find a sequence of decision 
functions \(\{\delta^*_1(k_0, w_0), \delta^*_2(k_1, w_1), \ldots, \delta^*_T(k_{T-1}, w_{T-1})\}\) that maximizes the total expected reward.
The optimal policy is obtained by solving recursively:
\[u_i(k_i, w_i) = \max \left\{ R_{i+1}((k_i, w_i), 0) + \sum_{j=0}^{n_i} p_{i+1} ((n_i r_i, w_i + j) \mid (k_i, w_i), 0) u_{i+1}(n_i r_i, w_i + j), \right\}
\[R_{i+1}((k_i, w_i), 1) \right\}

where \(u_i(k_i, w_i)\) represents the maximum expected reward earned by continuing optimally 
from state \((k_i, w_i)\) onwards. This reward is obtained when taking action \(a_i^*(k_i, w_i)\), which 
represents the optimal action \(a\) to take when in state \((k_i, w_i)\).

3.3. Characteristics of the optimal policy to be found.

In principle, the recursive equation can be solved backwards from period \(T\) to identify the 
optimal action for each possible state. However, depending on the length of the study 
horizon, the sales volumes, and the granularity of the state space (i.e., the definition of a 
“unit” sold or processed), the number of states to be evaluated could grow very large. To 
reduce the amount of computation as well as improving communication, appeal to decision-
makers, and managerial insight, it is desirable to identify that an optimal policy of a simple 
form exists. In particular, based on a partial ordering of the state space, below we establish 
the existence of an optimal monotone policy, which corresponds to a threshold in one of the
state variables, beyond which the outsourcing action $a = 1$ is optimal. Such a form also facilitates exploration of conditions under which outsourcing is more likely to be optimal. Below this threshold, the firm should continue performing the RL activities internally ($a = 0$). Next, conditions for the existence of an optimal deterministic nondecreasing policy will be defined in terms of the MDM proposed.

3.3.1. Conditions for identifying a monotone deterministic nondecreasing policy as optimal.

As stated by Puterman (1994), there exist sets of conditions that ensure that optimal policies are monotone in the system state. For such a concept to be meaningful, it is required that the state have a physical interpretation and some natural ordering. The expression “monotone policy” refers to a monotone deterministic Markovian policy.

For the MDM proposed, the states are partially ordered in terms of the cumulative returned units $w_t$. Specifically, for each $t$, let the states $(k_t, w_t)$ be strictly partially ordered according to the next criteria:

1. For every $t$, group the states where $k_t$ has a particular value (defined as $k_t^i, k_t^x, k_t^h \ldots$).

2. For each group generated, generate a logical ordering for the states according to $w_t$, i.e.; the larger $w_t$ is, the greater the state is.

This strict partial ordering implies that $(k^1, w^1) \prec (k^2, w^2) \iff k^1 = k^2 \ and \ w^1 < w^2$, which is illustrated in Figure 2.

Figure 2. Criterion followed for a strict partial state ordering.

In addition to the partial ordering defined, a cumulative probability also has to be defined in order to identify the conditions for a monotone nondecreasing policy:
\[ q_t[(k_i, w_i), (k_{i-1}, w_{i-1}), a] = \sum_{w_i=w_{i-1}}^{n_{i-1}} p_i[(k_i, w_i), (k_{i-1}, w_{i-1}), a], \]

where for \( a = 0 \) we have:

\[
q_t[(n_{i-1}r, w_i), (k_{i-1}, w_{i-1}), 0] = \begin{cases} 
\sum_{j=w_{i-1}}^{n_{i-1}} \binom{n_{i-1}}{j} r^j (1-r)^{n_{i-1}-j} & \text{for } w_i \geq w_{i-1} \\
1 & \text{for } w_i < w_{i-1}
\end{cases}
\]

and for \( a = 1 \) we have:

\[
q_t[(0, w_i), (k_{i-1}, w_{i-1}), 1] = \begin{cases} 
\sum_{j=w_{i-1}}^{n_{i-1}} \binom{n_{i-1}}{j} r^j (1-r)^{n_{i-1}-j} & \text{for } w_i \geq w_{i-1} \\
1 & \text{for } w_i < w_{i-1}
\end{cases}
\]

Finally, recall the definition of a superadditive function. Let \( X \) and \( Y \) be partially ordered sets and \( g(x, y) \) a real-valued function on \( X \times Y \). It is said that \( g \) is superadditive if for \( x^- \leq x^+ \text{ in } X \text{ and } y^- \leq y^+ \text{ in } Y \),

\[
g(x^+, y^+) + g(x^-, y^-) \geq g(x^+, y^-) + g(x^-, y^+).
\]

One set of conditions stated by Puterman (1994) for the existence of a monotone optimal policy are:

1. \( R_t((k, w), a) \) is nondecreasing in \((k, w)\) for \( a \in \{0,1\} \),
2. \( q_t[(k_i, w_i=w_j), (k, w), a] \) is nondecreasing in \((k, w)\) for all \( w_i \) and \( a \in \{0,1\} \),
3. \( R_t((k, w), a) \) is a superadditive function on \((k, w) \times a\),
4. \( q_t[(k_i, w_i=w_j), (k, w), a] \) is a superadditive function on \((k, w) \times a\), and
5. \( R_t(k_T, w_T) \) is nondecreasing in \((k_T, w_T)\).

When all of these conditions are satisfied, there exists a monotone nondecreasing policy that is optimal.

3.4 Requirements in the MDM for the existence of a Monotone Nondecreasing Policy
In order to prove these five conditions for the MDM developed, the next lemma will be used:

**Lemma 1.**

Suppose \( X_n \) is binomial with parameters \( n \) and \( r \), where \( n = 2, 3, \ldots \), and \( 0 < r < 1 \). Let \( \mu_n = E(X_n) = nr \). Then for any \( n \) and \( l = 1, 2, \ldots, n-1 \),

\[
E[(X_n - \mu_n)^+] \geq E[(X_i - \mu_i)^+]
\]

\[
E[\min(X_n, \mu_n)] \geq E[\min(X_i, \mu_i)]
\]

(c) For any integer \( m \) such that \( 0 \leq m \leq l \), \( P\{X_n > m\} \geq P\{X_i > m\} \).

**Proof:** See Appendix.

**Theorem 1**

If inequalities (7) - (11) are satisfied for the cost parameters and

\[
r \leq \left( \frac{c_5 - c_7}{c_5 - c_1 - c_3 - c_4} \right),
\]

then a monotone nondecreasing policy is optimal.

The proof is presented in the following five subsections:

3.4.1. Condition 1.

This condition holds when the cost of either action increases with the number of items sold but not yet returned. For \( a=1 \), it requires that \( R_i((k_{i-1}, w_{i-1}), 1) \leq R_i((k_{i-1}, w_{i-1} + i), 1) \) for \( 1 \leq i \leq n_{i-1} \), which follows immediately from \( i > 0 \) and \( c_7 > 0 \).

For \( a=0 \), the condition \( R_{i+1}((k_i, w_i), 0) \leq R_{i+1}((k_i, w_i + i), 0) \) for \( 1 \leq i \leq n_{i-1} \), i.e., that the expected internal RL reward is greater when the cumulative amount of returned units \( w_i \) is greater is equivalent to:

\[
-c_i \left( (n_r - k_i)^+ - (n_i - i)(r - k_i)^+ \right) - c_2 \left( (k_i - n_r)^+ - (k_i - (n_i - i)r)^+ \right) - c_3 r i - c_4 \left( E[\min(X, n_r)] - E[\min(Y, (n_i - i)r)] \right) - c_5 \left( E[(X - n_r)^+] - E[(Y - (n_i - i)r)^+] \right) \leq 0,
\]

\[1 \leq i \leq n_i\]
where \( c_1, c_3, c_4, c_5 > 0 \) and \( X (Y) \) is binomially distributed with parameters \( n_t \ (n_t - i) \), respectively, and \( r \). From parts (a) and (b) of Lemma 1, the elements that multiply \( c_4 \) and \( c_5 \) are nonnegative.

This inequality can be analyzed in three cases:

1) \( n_r < k_i \)
2) \( k_i \leq (n_t - i)r \)
3) \( (n_t - i)r \leq k_i \leq n_t r \)

In the first case, it is equivalent to:

\[
\begin{align*}
-ir(c_3 - c_2) - c_4 \left[ E \left[ \min(X, n_r) \right] - E \left[ \min(Y, (n_t - i)r) \right] \right] \\
- c_5 \left[ E \left[ (X - n_r)^+ \right] - E \left[ (Y - (n_t - i)r)^+ \right] \right] & \leq 0, \ 1 \leq i \leq n_t
\end{align*}
\]

which is satisfied, given inequality (12). In the second case, it can be reduced to:

\[
\begin{align*}
-ir(c_1 + c_3) - c_4 \left[ E \left[ \min(X, n_r) \right] - E \left[ \min(Y, (n_t - i)r) \right] \right] \\
- c_5 \left[ E \left[ (X - n_r)^+ \right] - E \left[ (Y - (n_t - i)r)^+ \right] \right] & \leq 0, \ 1 \leq i \leq n_t
\end{align*}
\]

which follows from the assumption of positive cost coefficients. Finally, for the last case it is:

\[
\begin{align*}
(k_t - n_r)(c_1 + c_3) - c_4 \left[ E \left[ \min(X, n_r) \right] - E \left[ \min(Y, (n_t - i)r) \right] \right] \\
- c_5 \left[ E \left[ (X - n_r)^+ \right] - E \left[ (Y - (n_t - i)r)^+ \right] \right] & \leq 0, \ 1 \leq i \leq n_t
\end{align*}
\]

which holds under inequalities (7) and (8).

3.4.2. Condition 2.

This condition holds when it is more likely to meet or exceed a given number of cumulative returns in the next period, if a higher number of returns have been experienced up to the current period. Then, this condition requires that for a fixed \( w_i = w_f \):

\[
q_i \left( (k_t, w_i) \mid (k_{t-1}, w_{t-1}), a \right) \leq q_i \left( (k_t, w_i) \mid (k_{t-1}, w_{t-1} + i), a \right), \quad 1 \leq i \leq n_{t-1},
\]

which can be analyzed under the three cases:
1) \( w_t \leq w_{t-1} \)
2) \( w_{t-1} < w_t \leq w_{t-1} + i \)
3) \( w_{t-1} + i < w_t \)

In the first case, the cumulative returns in period \( t-1 \) are greater or equal than \( w_t \). Then, the probability that such cumulative returns will equal or exceed \( w_t \) in the next period is 1; i.e., this condition is always satisfied as an equality in this case.

In the second case, the cumulative returns are already greater than \( w_t \) in the right hand side of the inequality \( (w_t \leq w_{t-1} + i) \). This implies that the probability on the right hand side equals 1, so that the inequality holds regardless of the probability on the left hand side.

Finally, for the third case, this condition can be rewritten as follows:

\[
\sum_{j=0}^{n_{t-1}-1} \binom{n_{t-1}}{j} r^j (1-r)^{n_{t-1}-j} \leq \sum_{j=0}^{n_{t-1}-1} \binom{n_{t-1} - i}{j} r^j (1-r)^{n_{t-1}-j-i}, \quad w_{t-1} < w_{t-1} + i < w_t,
\]

which can be stated as:

\[
\sum_{j=0}^{n_{t-1} - n_{t-1} - i} \binom{n_{t-1}}{j} r^j (1-r)^{n_{t-1}-j} \geq \sum_{j=0}^{n_{t-1} - n_{t-1} - i} \binom{n_{t-1} - i}{j} r^j (1-r)^{n_{t-1}-j-i},
\]

and is equivalent to:

\[
P(X \leq w_t - w_{t-1} - 1) \geq P(Y \leq w_t - w_{t-1} - 1 - i).
\]

This follows directly from Lemma 1(c).

3.4.3. Condition 3.

\[
R_{s+1}((k_s, w_t), 1) - R_{s+1}((k_s, w_t), 0) \leq R_{s+1}((k_s, w_t + i), 1) - R_{s+1}((k_s, w_t + i), 0)
\]

This inequality holds when for a fixed capacity \( k_s \), the incremental effect on the RL reward of switching to an outsourcing strategy is greater when the cumulative amount of returned units \( w_t \) is greater. In other words, given a fixed capacity \( k_s \), the difference between the internal and outsourcing RL cost is greater when the current cumulative returned amount of units is greater. This condition can be rewritten as:
where, as in Condition 1, $X (Y)$ is binomially distributed with parameters $n_t (n_t - i)$, respectively, and $r$, and from Lemma 1 (a) and (b), the expressions that multiply $c_4$ and $c_5$ are nonnegative.

Consider the three cases:

1) $n_r < k_i$
2) $k_i \leq (n_i - i) r$
3) $(n_i - i) r \leq k_i \leq n_i r$

In the first case, the inequality is reduced to:

$$c_i ((n_i - i)r - k_i)^+ + c_2 ((k_i - (n_i - i)r)^+ (k_i - n_i r)^+) + ic_i - irc_i \geq c_4 E[min(X,n_i r)] - E[min(Y,(n_i - i)r)] + c_5 E[(X - n_i r)^+] - E[(Y - (n_i - i)r)^+]$$

for $1 \leq i \leq n_i$

Considering that $X = Y + \sum_{j=n_i-i+1}^{n_i} U_j$, where each $U_j$ is independently 1 with probability $r$ and 0 otherwise, this inequality can be analyzed under the four possible cases shown in Table 1.

Table 1. Cases for Condition 3 when $n_i r < k_i$

In the second case the inequality is reduced to:

$$c_i ((n_i - i)r - k_i)^+ + c_2 ((k_i - (n_i - i)r)^+ (k_i - n_i r)^+) + ic_i - irc_i \geq c_4 E[min(X,n_i r)] - E[min(Y,(n_i - i)r)] + c_5 E[(X - n_i r)^+] - E[(Y - (n_i - i)r)^+]$$

Following the same analysis, the resulting inequalities in the worst cases are shown in Table 2.

Table 2. Cases for Condition 3 when $k_i \leq (n_i - i)r$
Finally, in the third case the inequality is reduced to:

\[
c_i - c_i i r + c_1(n_i r - k_i) + c_2(k_i - (n_i - i) r) \geq c_4 \left[ E \left[ \min \left( X, n_i r \right) \right] - E \left[ \min \left( Y, (n_i - i) r \right) \right] \right] + \\
c_3 \left[ E \left[ (X - n_i r)^+ \right] - E \left[ (Y - (n_i - i) r)^+ \right] \right]
\]

By comparison with inequality (12), this inequality will be satisfied as long as:

\[
(c_1 - c_2)(n_i r - k_i) \geq 0
\]

which is true given inequality (7) and that \( n_i r \geq k_i \) in this case.

Then, the following relationships are required to satisfy Condition 3:

\[
c_7 - c_5 \geq (c_3 - c_2) r - (c_5 - c_4) r \\
(13)
\]

\[
c_7 - c_4 \geq (c_3 - c_2) r \\
(14)
\]

\[
c_7 - c_3 \geq (c_1 + c_3) r - (c_3 - c_4) r \\
(15)
\]

\[
c_7 - c_4 \geq (c_1 + c_3) r \\
(16)
\]

However, inequalities (15) and (16) are redundant. Also, given that \( (c_5 - c_4) \geq (c_3 - c_4) r \), then (13) and (14) can be reduced to:

\[
r \leq \frac{c_5 - c_7}{c_5 - c_1 - c_3 - c_4}. \\
(17)
\]

This represents an upper limit on the return rate \( r \), and the required inequality to satisfy Condition 3 (in addition to the assumptions on the cost parameters stated on Section 3.1.5).

3.4.4. Condition 4.

This condition implies that the difference between the cumulative probability that returns exceed a given number when taking the outsourcing option and when performing RL activities internally, is greater when the current returns are greater. This condition can be written as follows:
for \( w_{t+1} > w_{t+1}' \). Given that such transition probability values do not depend on the current RL capacity \((k_t)\), which is changed when the outsourcing decision is taken, this condition is satisfied as an equality.

3.4.5. Condition 5.

This condition implies that the terminal reward is greater when the amount of cumulative returns is greater. The inequality \( R(T, k_T, w_T) \leq R(T, k_T, w_T + i) \) for \( 1 \leq i \leq n_T \) can be written as

\[
c_i k_T - c_5 n_T \leq c_i k_T - c_5 (n_T - i) r
\]

which follows from \( c_5 > 0 \).

3.4.6. Corollaries and implications.

**Corollary 1**

If inequalities (7) to (11) are satisfied, and also:

\[
c_7 \leq c_1 + c_3 + c_4
\]

Then there is an optimal monotone nondecreasing policy for any \( r \leq 1 \).

**Corollary 2**

If inequalities (7) to (11) are satisfied and also:

\[
c_5 - c_7 \geq c_7 - (c_1 + c_3 + c_4)
\]

Then there is an optimal monotone nondecreasing policy for any \( r \leq 0.5 \).

Inequality (18) implies that the unit cost of outsourcing RL functions is less than or equal to the corresponding unit capacity cost of creating and keeping enough capacity to remanufacture one unit, including its reprocessing cost; i.e., the unit cost of developing capacity and remanufacturing returns internally is greater than the unit outsourcing cost. If
such a situation takes place, then there exists an optimal monotone nondecreasing policy, regardless of the value that the return rate takes.

On the other hand, inequality (19) implies that the opportunity (regret) cost $c_7 - c_7$ of not taking the outsourcing option and incurring the corresponding shortage for a particular unit, is greater than the opportunity cost $c_7 - (c_1 + c_3 + c_4)$ of taking the outsourcing option, instead of creating and keeping internal capacity to remanufacture that unit; i.e., considering that (as mentioned in section 3.1.5) the 3PRLP has infinite capacity, the regret of incurring a shortage when the outsourcing option was not taken, is greater than the regret of incurring the outsourcing cost instead of creating and using internal capacity. If such a situation takes place, then there exists an optimal monotone nondecreasing policy for any RL system where the return rate is below 0.5.

The previous results, as well as the fact that the return rate in most industries is between zero and 0.3, approaching 0.5 only in some specific sectors (Rogers and Tibben-Lembke, 1999), imply that the cases where $r \leq 0.5$ and inequality (19) is not satisfied are of special interest. In other words, there is no certainty about the existence of an optimal monotone nondecreasing policy in such cases.

The result of Theorem 1 implies that, in any period $t$:

$$ k^1_t = k^2_t \quad \text{and} \quad w^1_t < w^2_t \Rightarrow a^*_t \left( k^1_t, w^1_t \right) \leq a^*_t \left( k^2_t, w^2_t \right). $$

In the remainder of this section, the subscript $t$ is suppressed for simplicity. Define the outsourcing threshold for each capacity level as:

$$ \theta(k) = \begin{cases} \min \{ w : a^*(k, w) = 1 \} & \text{if } \theta(k) \text{ exists} \\ \infty & \text{otherwise} \end{cases} $$
**Lemma 2:**
Suppose conditions (7) and (8) are satisfied. Let $\theta(k;r)$ be the value of $\theta(k)$ when the return rate is $r$. If $0 \leq r \leq r + \Delta_r \leq 0.5$ and
\[ c_5 \leq 4.375c_4, \]  \hspace{1cm} (20)
then
\[ \theta(k;r) \geq \theta(k;r + \Delta_r) \]  \hspace{1cm} (21)

**Proof:** In the Appendix.

Note that the maximum ratio $c_5/c_4$ in (20) is a lower bound on the requirement that holds for states with any number of outstanding items, $n \geq 2$. In a practical situation, when $n$ would be much larger, the inequality imposes no significant constraint on $c_5/c_4$.

**Lemma 3:**
Let $q((nr, w_i)(k, w);0;r)$ be the value of $q((nr, w_i)(k, w);0)$ when the return rate is $r$. If $0 \leq r + \Delta_r \leq 0.5$ then:
\[ q((n(r + \Delta_r), w_i)(k, w);0;r + \Delta_r) \geq q((nr, w_i)(k, w);0;r) \]  \hspace{1cm} (22)

**Proof:** In the Appendix.

**Theorem 2:**

*Suppose the conditions of Theorem 1 and Lemma 2 are satisfied. Then*
\[ q[(nr, \theta(k;r + \Delta_r))(k, w);0;r + \Delta_r] \geq q[(nr, \theta(k;r))(k, w);0;r]. \]

**Proof:**

From Lemma 2 we have $\theta(k;r) \geq \theta(k;r + \Delta_r)$, which implies that:
\[ q[(nr, \theta(k;r + \Delta_r))(k, w);0;r] \geq q[(nr, \theta(k;r))(k, w);r]. \]  \hspace{1cm} (23)
From Lemma 3:

\[ q\left[(n(r+\Delta_r), \theta(k, r+\Delta_r))(k, w), 0; r+\Delta_r \right] \geq q\left[(nr, \theta(k, r))(k, w), r \right]. \tag{24} \]

Then, we have by the transitive property that:

\[ q\left[(n(r+\Delta_r), \theta(k, r+\Delta_r))(k, w), 0; r+\Delta_r \right] \geq q\left[(nr, \theta(k, r))(k, w), r \right], \]

which completes the proof.

Theorem 2 implies that the suitability of the outsourcing option increases when the return rate increases. This comes not only from the fact that the probability of crossing the corresponding threshold that determines the optimality of the outsourcing option increases, but also from the fact that the value for such threshold does not increase.

Then, given that the variability on the return volume increases as the return rate increases (for any value below 0.5), it can be concluded that outsourcing becomes a more suitable option for products with greater variability in their return volume.

This conclusion is supported by the fact that (as shown in Lemma 2) in most cases, the expected reward for \( a = 0 \) decreases as the return rate, and in consequence the variability in the return volume, increases. This decrease for the reward when performing RL activities internally causes the threshold that determines the optimality of the outsourcing option not to increase. Moreover, as the return rate increases, the threshold may decrease, which expands the set of states where outsourcing is optimal. This situation will be supported by two numerical examples that will be shown in the next section.

3.5. Numerical examples.

In order to show the influence of higher variability of the return volume on the suitability of an outsourcing option, consider a particular scenario defined by the parameters:
As well as the sales function:

\[
S_t = \begin{cases} 
\frac{2M}{L} t, & t = 1, 2, \ldots, L/2 \\
M - \frac{2M}{L} (t-L/2-1), & t = L/2 + 1, \ldots, L 
\end{cases}
\]

(26)

where \(M = 3\). The values for the cost parameters satisfy conditions (7) to (11) as well as (18); i.e., there is an optimal monotone nondecreasing policy for any \(r \leq 1\).

Table 3 shows the values for the threshold in each set of states, for \(r = \{0.2, 0.3, 0.4, 0.5\}\), as well as the probability \(q_t(k_i, s_i | k_{i-1}, s_{i-1}, a)\) that such threshold (defined as \(w_i\)) is crossed in each case. These values were obtained by creating a Matlab program, whose inputs are \(r, L, W, c_1, c_2, c_3, c_4, c_5, c_6, c_7\), as well as the sales volume \(s_t\) during the analysis horizon. Based on this information, the program computes the possible states and orders them according to the criteria defined. The program also computes the amount of units \(n_t\) outstanding in the market for each state, as well as the corresponding transition probabilities and expected costs for \(a = 0\) and \(a = 1\). The terminal costs are also obtained. Based on this, the program solves the MDM by using backward induction, and shows the optimal action to take at each decision epoch.

Table 3. Value of the threshold and the probability of crossing it for \(r = \{0.2, 0.3, 0.4, 0.5\}\)
As it can be identified in Table 3, a greater variability in the return volume (greater $r$) increases the probability of crossing the corresponding threshold in each set of states; i.e., there is a greater probability that outsourcing ($a = 1$) will be the optimal action to take.

This implies that, as mentioned in section 2.2, greater variability in the return volume increases the uncertainty about the volume of units put into the corresponding RL system, which forces the firm to follow an outsourcing strategy, and take advantage of the economies of scale by involving a 3PRLP in managing returned items.

Now, in order to show the influence of higher variability in the return volume as well as a shorter product’s life cycle, consider the next two cases:

\[
\begin{align*}
\text{Case 1:} & \quad r = 0.3, \quad L = 5 \\
\text{Case 2:} & \quad r = 0.5, \quad L = 4
\end{align*}
\] (27)

where both of them are defined by the cost parameters and value for $W$ shown in (25), as well as sales function (26) with $M = 3$. Tables 4 and 5 show the results for the two cases considered.

Table 4. Value of the threshold and the probability of crossing it for Case 1: $r = 0.3, \quad L = 5$

Table 5. Value of the threshold and the probability of crossing it for Case 2: $r = 0.5, \quad L = 4$

By comparing both cases, it can be identified that when the variability in the return volume increases and the length of the life cycle decreases, the probability of crossing the threshold beyond which outsourcing is optimal increases.

These examples with $r \leq 0.5$ illustrate how the threshold that determines the suitability of the outsourcing option for the Markov Decision Model developed is easily crossed in the scenario where the variability on the return volume is greater (greater $r$), and
the length $L$ of the product life cycle is shorter. This implies that, due to a high variability in
the rate of returns, it may not be economically feasible for a firm to develop its own RL
facilities, given that the amount of units to be returned will be significantly uncertain over
time, and the required capacity will be changing constantly.

The complexity of this situation increases when the life cycle for this type of products
is extremely short, which requires quick but adequate decisions for these RL systems, in
order to efficiently respond to such changing conditions. This can effectively be
accomplished by involving a 3PRLP, which specializes in these activities, and can take
advantage of the economies of scale to convert RL functions into a profit-creating activity in
the closed-loop chain.

4. Conclusions and Future Work

A Markov Decision Model (MDM) for evaluating an outsourcing option in RL is developed
in this research. It considers several elements that are critical in defining the characteristics of
a RL network, such as the uncertainty in the return volume, the length of the product life
cycle, the sales behavior, the particular RL costs incurred, as well as the length of time
defined for the existence of that RL system. In particular, the length of the product life cycle,
the cost parameters, the sales function defined and the rate of return considered, are modeling
the scenario of interest; i.e., the length of the horizon analysis in the MDM is determined by
such life cycle; while the uncertainty implied in the MDM is represented by the expected
amount of returned units, which is defined by the outstanding units in the market and the rate
of return considered.

The conditions for the existence of an optimal monotone nondecreasing policy were
also shown, where it was verified that such a policy will exist as long as a set of
nonrestrictive assumptions on the cost parameters is satisfied, and the return rate is below a
bound defined in terms of those cost parameters. Moreover, there are some instances where an optimal monotone nondecreasing policy exists, regardless of the value for the return rate.

The existence of an optimal monotone nondecreasing policy implies the presence of a threshold above which it is optimal to follow an outsourcing strategy for the RL system; otherwise, to continue performing the RL activities internally. This threshold was defined in terms of a partial ordering for the system states, where given a fixed capacity at a decision epoch, the states are ordered according to the cumulative returned units, such that if that volume goes above a particular level, then it is optimal to follow an outsourcing strategy and take advantage of the economies of scale implied by involving a 3PRLP in managing the returns, which has RL as its core function.

It was also shown that outsourcing is a more suitable option for scenarios with greater variability on the return volume, by explaining analytically the increment in the probability of crossing the threshold that determines outsourcing optimality when the variability in the return volume increases. It was also shown how the threshold does not increase when the return volume variability increases. It may even decrease as such variability increases, which also increases the probability of crossing it.

As a support to this analysis, two sets of scenarios were explored numerically. In the first set, the rate of returns was increased while keeping everything else fixed. The results showed that outsourcing is more suitable when the rate of returns (and in consequence the variability in the return volume) is greater. The second set contained two different scenarios with the same cost parameters and sales function, but with different variability in the return volume and length of the product’s life cycle. In the second scenario, a greater variability and shorter life cycle existed, and (as expected) outsourcing was a more suitable option in this scenario than in the first one.
4.2. Future Work

Even though the existence of an optimal monotone nondecreasing policy was proved, the influence on the suitability of the outsourcing option was analytically proved only for the return volume variability, but not for the length of the product life cycle, whose influence was shown merely numerically. Developing an analytical proof for the influence of the life cycle length on outsourcing suitability represents a future research area to consider. The main challenge for this analysis is the difference in the cardinality of the sets of ordered states obtained for each case. This comes from the fact that the size of the state space at each decision epoch is determined by the sales function of the product analyzed, as well as the length of the lifecycle.

Finally, future research can also consider various extensions to this model. Several problem parameters, such as the rate of returns and/or the RL costs may not be constant during the product’s life cycle. Nonstationary costs will be easy to incorporate, but variation in the return rate will require more elaborate modifications to the analysis. More generally, another area of research would identify the requirements for the existence of an optimal monotone nondecreasing policy, when the returns follow a probability distribution different than the one described in this paper. The influence of a different stochastic behavior for the returns (according to a particular scenario of interest) can be considered, which will represent the basis for evaluating the five conditions required for such a policy structure.

A full analysis of the outsourcing decision should also consider the possibility that internal management does not imply that the RL capacity is adjusted to the expected returns, either because the firm does not have the capability of adjusting its capacity each period, or because a different adjustment policy is found to result in better performance. The irreversibility of the outsourcing decision assumed in this paper also could be relaxed, in view of the fact that the 3PRLP selected may fail to perform adequately. Finally, the potential
for profits from reprocessing and selling returned items may be considered as a benefit of maintaining RL capacity internally.

References


**APPENDIX**

**Proof of Lemma 1.**

(a) Suppose \( l = n - 1 \) and consider

\[
E \left[ (X_n - nr)^+ - (X_{n-1} - (nr - r))^+ \right] = E \left[ E \left[ (X_n - nr)^+ - (X_{n-1} - (nr - r))^+ \mid X_{n-1} \right] \right]
\]

where \( X_n = X_{n-1} + U \)

where \( U = 1 \) with probability \( r \) and 0 otherwise. There are three possible cases for \( X_{n-1} = m \), which are shown in Table 6.

<table>
<thead>
<tr>
<th>Case Considered for ( X_{n-1} = m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the first case, ( E \left[ (X_n - nr)^+ - (X_{n-1} - (nr - r))^+ \right] = r(m + 1 - nr)^+ ) given that ( nr &gt; m ). Then, this case yields a nonnegative result. In the second case, ( E \left[ (X_n - nr)^+ - (X_{n-1} - (nr - r))^+ \right] = (nr - m)(1 - r) ) where, given that ( nr &gt; m ), the expression is also nonnegative. Finally, for the third case, ( E \left[ (X_n - nr)^+ - E \left[ (X_i - (nr - r))^+ \right] \right] = 0 ), given that ( m \geq nr ). Then, to complete the conditioning argument:</td>
</tr>
</tbody>
</table>
Then, considering that \( E\left( X_n - \mu_n \right)^+ \geq E\left( X_{n-1} - \mu_{n-1} \right)^+ \) and, since \( n \) is arbitrary, we have, by the transitive property:

\[
E\left( X_n - \mu_n \right)^+ - E\left( X_{i} - \mu_{i} \right)^+ \geq 0 \quad \text{for any} \quad l = 1, 2, \ldots, n-1
\]

which completes the proof.

(b) Suppose \( l = n-1 \) and consider

\[
E\left[ \min(X_n,nr) - \min(X_{n-1},(nr-r)) \right] = E\left[ E\left[ \min(X_n,nr) - \min(X_{n-1},(nr-r)) \mid X_{n-1} \right] \right]
\]

where \( X_n = X_{n-1} + U \)

where \( U = 1 \) with probability \( r \) and 0 otherwise. Suppose \( X_{n-1} = m < nr - r \). Then

\[
E\left[ \min(X_n,nr) - \min(X_{n-1},(nr-r)) \mid X_{n-1} = m \right] = r \min(m+1,nr) + (1-r)m - m
\]

since \( m < nr - ir \Rightarrow m < nr \). Also, \( nr > m + r \Rightarrow \min(m+1,nr) \geq m + r \), so

\[
 r \min(m+1,nr) + (1-r)m - m \geq r(m+r) + (1-r)m - m = r^2.
\]

On the other hand, suppose \( X_{n-1} = m \geq nr - r \). Then

\[
E\left[ \min(X_n,nr) - \min(X_{n-1},(nr-r)) \mid X_{n-1} = m \right] = nr + (1-r)\min(m,nr) - nr + r
\]

because \( m \geq nr - r \Rightarrow m + 1 \geq nr \). Also, \( \min(m,nr) \geq nr - r \), which implies that

\[
rnr + (1-r)\min(m,nr) - nr + r \geq nr^2 + (1-r)(nr-r) - nr + r = r^2.
\]

To complete the conditioning argument:

\[
E\left[ \min(X_n,nr) - \min(X_i,nr-r) \right] = E\left[ E\left[ \min(X_n,nr) - \min(X_i,nr-r) \mid X_i \right] \right]
\]

\[
= \sum_{m=0}^{n-1} E\left[ \min(X_n,nr) - \min(X_i,nr-r) \mid X_i = m \right] P(X_i = m) \geq 0
\]
Then:

\[ E\left[\min(X_n, \mu_n)\right] - E\left[\min(X_{n-1}, \mu_{n-1})\right] \geq 0. \]

and since \( n \) is arbitrary, we have, by the transitive property:

\[ E\left[\min(X_n, \mu_n)\right] - E\left[\min(X_i, \mu_i)\right] \geq 0 \text{ for any } l = 1, 2, \ldots, n - 1 \]

which completes the proof.

(c) Suppose \( l = n - 1 \) and consider

\[
P\{X_n > m\} - P\{X_{n-1} > m\} = \left[ P\{X_{n-1} > m\} + P\{X_{n-1} = m\} P\{U = 1\} \right] - P\{X_{n-1} > m\}
\]

\[
= \left( \begin{array}{c} n - 1 \\ m \end{array} \right) r^{-m} (1 - r)^{n-1-m} > 0, \quad \text{where } X_n = X_{n-1} + U
\]

which implies that the probability of experiencing more than \( m \) successes is greater when one additional trial is added to the sequence of Bernoulli trials. Then, since \( n \) is arbitrary, we have, by the transitive property:

\[ P\{X_n > k\} - P\{X_i > k\} > 0 \quad \text{for any } l = 1, 2, \ldots, n - 1 \]

which completes the proof.

**Proof of Lemma 2:**

Let \( w = \theta(k; r) \), and let \( R((k, w), a; r) \) be the value of \( R((k, w), a) \) when the return rate is \( r \). In order to prove (24), the next inequalities must be satisfied, given the relationships for \( r \) and \( \Delta_r \) defined in terms of the cost parameters:

\[
R((k, w), 0; r) \leq R((k, w), 1; r)
\]  \hspace{1cm} (28)

\[
R((k, w), 0; r + \Delta_r) \leq R((k, w), 1; r + \Delta_r)
\]  \hspace{1cm} (29)

where (28) comes from the definition of \( w \). Inequality (29) implies that the threshold is not greater than \( w \) when the return rate increases \( \Delta_r \); i.e., the threshold does not increase when the return rate increases, as stated in (21).
Given that by definition of \( w \), inequality (28) is satisfied, and that the right-hand sides in both inequalities are equal (they do not depend on \( r \)), inequality (29) will be satisfied as long as:

\[
R((k,w)_0; r + \Delta_r) \leq R((k,w)_0; r),
\]

where (30) can be rewritten as:

\[
c_1((n(r + \Delta_r) - k) - (nr - k)) + c_2((k - n(r + \Delta_r)) - (k - nr)) + c_j n \Delta_r + c_4 E\left(\min\left(x_1, n(r + \Delta_r)\right)\right) - E\left(\min\left(x_2, nr\right)\right) + c_5 E\left(\max\left(x_1 - n(r + \Delta_r), 0\right)\right) - E\left(\max\left(x_2 - nr, 0\right)\right) > 0 \quad \text{where} \quad x_1 \sim \text{Bin}(n, r + \Delta_r),
\]

\[
x_2 \sim \text{Bin}(n, r)
\]

where, given (7) and (8), the following part of (31) is nonnegative:

\[
c_1((n(r + \Delta_r) - k) - (nr - k)) + c_2((k - n(r + \Delta_r)) - (k - nr)) + c_j n \Delta_r \geq 0
\]

Consider the rest of the elements of (31):

\[
c_4 E\left(\min(x_1, n(r + \Delta_r))\right) - E\left(\min(x_2, nr)\right) + c_5 E\left(\max(x_1 - n(r + \Delta_r), 0)\right) - E\left(\max(x_2 - nr, 0)\right)
\]

Let \( g_n(r) = E\left[\min(X, nr)\right] \) and \( f_n(r) = E\left[\max(X - nr, 0)\right] \) where \( X \sim \text{binomial}(n, r) \) and \( n \geq 2 \). Then, (32) will be true if the combined cost function \( c_2 g_n(r) + c_j f_n(r) \) is an increasing function of \( r \).

The derivatives of \( f_n \) and \( g_n \) are discontinuous at \( r = j/n, j = 1, \ldots, n-1 \), which implies that a separate expression is needed for the derivatives in each interval for \( r \). For \( \frac{j}{n} < r \leq \frac{j+1}{n} \), or equivalently \( j < nr \leq j + 1 \), where \( 2j < n \):

\[
f_n(r) = \sum_{k=j+1}^{n} (k - nr) p(k) = \sum_{k=0}^{\frac{j}{n}-1} (nr - k) p(k)
\]

and

\[
g_n(r) = \sum_{k=1}^{j} kp(k) + nr \sum_{k=j+1}^{n} p(k) = nr - \sum_{k=0}^{\frac{j}{n}-1} (nr - k) p(k)
\]
where $p(k) = \binom{n}{k} r^k (1-r)^{n-k}$. Then

$$c_4 g_n(r) + c_5 f_n(r) = c_4 \left( nr - \sum_{k=0}^{j} (nr - k) p(k) \right) + c_5 \left( \sum_{k=0}^{j} (nr - k) p(k) \right) = c_4 nr + (c_5 - c_4) \sum_{k=0}^{j} (nr - k) p(k).$$

We wish to show that $\frac{\partial}{\partial r} \left( c_4 g_n(r) + c_5 f_n(r) \right) \geq 0$ for reasonable values of $c_4 < c_5$.

(a) $\frac{\partial}{\partial r} \sum_{k=0}^{j} (nr - k) \binom{n}{k} r^k (1-r)^{n-k} = \binom{n}{j} (n-j) r^j (1-r)^{n-j-1} \left[ j + 1 - (n+1)r \right]$.

The proof is inductive, using integration by parts. First, for $j = 0$, or equivalently, $0 < nr \leq 1$,

integrating by parts with $u = 1 - (n+1)r$ and $dv = n(1-r)^{n-1} dr$ we get:

$$\int n(1-r)^{n-1} \left[ 1 - (n+1)r \right] dr = -\left[ 1 - (n+1)r \right] (1-r)^n - \int (n+1)(1-r)^n dr$$
$$= -\left[ 1 - (n+1)r \right] (1-r)^n + (1-r)^{n+1} = nr(1-r)^n.$$

And for $j = 1$, or equivalently, $1 < nr \leq 2$,

$$n \int (n-1)(1-r)^{n-2} \left[ 2r - (n+1)r^2 \right] dr = -n \left[ 2r - (n+1)r^2 \right] (1-r)^{n-1} + 2 \int n(1-r)^{n-1} \left[ 1 - (n+1)r \right] dr$$
$$= -nr \left[ 1 - nr + 1 - r \right] (1-r)^{n-1} + 2nr(1-r)^n$$
$$= nr(1-r)^n + (nr - 1)nr(1-r)^{n-1},$$

where the second equality results from substituting for the $j = 0$ integral.

Now, for $j > 1$, assume:

$$\binom{n}{j-1} \int (n-j+1)(1-r)^{n-j-1} \left[ jr^{j-1} - (n+1)jr^j \right] dr = \sum_{k=0}^{j-1} (nr - k) \binom{n}{k} r^k (1-r)^{n-k}.$$
&n j\int_0^1 (n-j)(1-r)^{n-j-1} [(j+1)r^j - (n+1)r^{j+1}]dr
\nonumber
&= -\left(\begin{array}{c}n \\ j \end{array}\right) r^j \left[ (j+1) - (n+1)r \right] (1-r)^{n-j} + (j+1) \left(\begin{array}{c}n \\ j \end{array}\right) \int_0^1 (1-r)^{n-j-1} [jr^{j-1} - (n+1)r^j] dr
\nonumber
&= -\left(\begin{array}{c}n \\ j \end{array}\right) r^j \left[ j-nr+1-r \right] (1-r)^{n-j} + \frac{j+1}{j} \left(\begin{array}{c}n \\ j-1 \end{array}\right) \int_0^1 (n-j+1)(1-r)^{n-j} [jr^{j-1} - (n+1)r^j] dr
\nonumber
&= \left(\begin{array}{c}n \\ j \end{array}\right) (nr-j) r^j (1-r)^{n-j} - \left(\begin{array}{c}n \\ j \end{array}\right) r^j (1-r)^{n-j+1} + \frac{j+1}{j} \sum_{k=0}^{j-1} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{n-k}
\nonumber
&= \sum_{k=0}^{j} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{n-k}, \text{ if } j \left(\begin{array}{c}n \\ j \end{array}\right) r^j = \sum_{k=0}^{j-1} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{j-k}. 
\nonumber

(b) To show: \( j \left(\begin{array}{c}n \\ j \end{array}\right) r^j = \sum_{k=0}^{j-1} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{j-k}. \) This is equivalent to:
\[ n(n-1)\cdots(n-j+1)r^j = (j-1)! \sum_{k=0}^{j-1} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{j-k} \]
and can be verified for \( j = 1 \) and \( j = 2. \) Then for \( j > 2, \) assume the equality is true for \( j - 1 \) as above. For \( j, \)
\[ j! \sum_{k=0}^{j} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{j-k} = j! \left[ \sum_{k=0}^{j-1} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{j-k} + (nr-j) \left(\begin{array}{c}n \\ j \end{array}\right) r^j \right] \]
\[ = j!(1-r) \sum_{k=0}^{j-1} (nr-k) \left(\begin{array}{c}n \\ k \end{array}\right) r^k (1-r)^{j-k} + (nr-j)n(n-1)\cdots(n-j+1)r^j \]
\[ = j!(1-r)n(n-1)\cdots(n-j+1)r^j + (nr-j)n(n-1)\cdots(n-j+1)r^j \]
\[ = n(n-1)\cdots(n-j)r^{j+1} \]
This completes the proof of (a).

(c) Using (a), for \( \frac{j}{n} < r \leq \frac{j+1}{n} \), or equivalently \( j < nr \leq j+1, \)
\[ \frac{\partial}{\partial r} \left[ c_4g_n(r) + c_5f_n(r) \right] = \frac{\partial}{\partial r} \left[ c_4nr + (c_5 - c_4) \sum_{k=0}^{j} (nr-k)p(k) \right] \]
\[ = c_4n + (c_5 - c_4) \left(\begin{array}{c}n \\ j \end{array}\right) (n-j)r^j (1-r)^{n-j-1} [j+1-(n+1)r] = c_4n + (c_5 - c_4) \phi(j,n,r). \]
Now, $\phi(j, n, r) < 0$ if $r > \frac{j+1}{n+1}$. Consider

$$\frac{\partial}{\partial r} \left[ (1-r)^{n-j-1} \left[ (j+1)r' - (n+1)r^j \right] \right] = r^{j-1} (1-r)^{n-j-2} \left[ n(n+1)r^2 - 2n(j+1)r + (j+1)f \right].$$

This quantity is negative between the two roots $r = \frac{j+1}{n+1} \pm \sqrt{\frac{n(j+1)(n-j)}{n(n+1)}}$. Clearly, the lower root is less than $\frac{j+1}{n+1}$, and it can be verified that the upper root is greater than or equal to $\frac{j+1}{n}$ (which is the upper endpoint of the interval for $r$ where this expression for the combined cost function is valid) since $n \geq 2$. Therefore, $\phi(j, n, r)$ takes its most negative value at $r = \frac{j+1}{n}$, where it equals:

$$-\left(\begin{array}{c} n \\ j \end{array}\right)(n-j)(j+1)\left(1-\frac{j+1}{n}\right)^{n-j-1}\left[j+1-(n+1)\frac{j+1}{n}\right]$$

$$=-\left(\begin{array}{c} n \\ j \end{array}\right)(n-j)(n-j-1)^{n-j-1}(j+1)^{j+1}$$

(d) Finally,

$$\frac{\partial}{\partial r} \left[ c_4 g_n(r) + c_5 f_n(r) \right] \geq 0 \text{ if } c_4 n \geq (c_5 - c_4) \left(\begin{array}{c} n \\ j \end{array}\right)(n-j)(n-j-1)^{n-j-1}(j+1)^{j+1}$$

or $c_5 \leq c_4 \frac{\left(\begin{array}{c} n \\ j \end{array}\right)(n-j)(n-j-1)^{n-j-1}(j+1)^{j+1}}{1+n^{j+1}\left[\left(\begin{array}{c} n \\ j \end{array}\right)(n-j)(n-j-1)^{n-j-1}(j+1)^{j+1}\right]} = c_4 \left[1+n^{j+1}\left[\left(\begin{array}{c} n \\ j \end{array}\right)(n-j)(n-j-1)^{n-j-1}(j+1)^{j+1}\right]\right] = c_4 \left[1+h(j, n)\right]$.

Because $h(j, n)$ is an increasing function of $n$, then for a fixed value of $j$, $c_5 \leq c_4 \left[1+h(j, 2j+1)\right]$ suffices for all $n > 2j$. In turn, $h(j, 2j+1) \geq h(1, 3) = 3.375$. Therefore, the expression in (32) is nonnegative as long as $c_5 \leq 4.375c_4$.  

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This completes the proof.

**Proof of Lemma 3:**

Inequality (22) can be rewritten as:

\[
\sum_{j=w_l}^{n} \binom{n}{j} (r + \Delta_j)(1 - r - \Delta_j)^{n-j} \geq \sum_{j=w_l}^{n} \binom{n}{j} r^j (1 - r)^{n-j}
\]

which is equivalent to:

\[
P\{x_1 > w_l - 1\} \geq P\{x_2 > w_l - 1\} \text{ for } w_l = \{1,2,...,n\}
\]

(33)

where \(x_1\) is binomial with parameters \(n\) and \(r + \Delta\) and \(x_2\) is binomial with parameters \(n\) and \(r\). The result follows from the fact that the family of binomial distributions for fixed \(n\) is stochastically increasing in \(r\) (Shaked and Shanthikumar, 1994).
Figure 1.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Production</th>
<th>Supply</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Collection</th>
<th>Selection</th>
<th>Re-Processing</th>
<th>Re-Distribution</th>
</tr>
</thead>
</table>

RL Facilities

- Which capacity level is optimal?
- How to adjust such capacity?
- Known variable (at a certain degree)

RL as core activity = no shortages, manages all future returns

3PRLP

- Third Party Reverse Logistics Provider Facilities
  - Flow of goods in RL chain
  - Flow of goods in "forward" chain

Unknown variable

- a=0
- a=1

Dispose RL firm's capacity

- c_1, c_2, c_3
- c_4
- c_5
- c_6
- c_7

Flow of goods in "forward" chain

Disposal

- Shortages
- c_5

Which happens if returns are greater than capacity developed? = shortages

3PRLP

- RL as core activity = no shortages, manages all future returns

- Flow of goods in RL chain

- c_1, c_2, c_3

- c_4

- c_5

- c_6

- c_7
Figure 2.

O = States grouped for $k_t$ and ordered according to $w_t$. Recall that some states may have several predecessors.
<table>
<thead>
<tr>
<th>Case</th>
<th>Value on the right-hand side of the inequality</th>
<th>Resulting inequality in the worst case:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1) ( X &gt; n_r, \ Y &gt; (n_r - i)r )</td>
<td>( c_4 r + c_5 (X - Y - ir) )</td>
<td>( c_7 - c_5 \geq (c_3 - c_2)r - (c_3 - c_4)r )</td>
</tr>
<tr>
<td>Worst case:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X - Y = i )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2) ( X &gt; n_r, \ Y \leq (n_r - i)r )</td>
<td>( n_r(c_4 - c_3) + c_4 X - c_4 Y )</td>
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</tr>
<tr>
<td>Worst case:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Y = nr - ir, )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X = nr - ir + i )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Because ( c_5 \geq c_4 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.3) ( X \leq n_r, \ Y \leq (n_r - i)r )</td>
<td>( c_4 (X - Y) )</td>
<td>( c_7 - c_4 \geq (c_3 - c_2)r )</td>
</tr>
<tr>
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<td></td>
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<tr>
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<td></td>
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<tr>
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<td>Worst case: $X - Y = i$</td>
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<tr>
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<tr>
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<td>Worst case: $Y = nr - ir$, $X = nr - ir + i$</td>
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</tr>
<tr>
<td></td>
<td>Because $c_4 &lt; c_5$</td>
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<td></td>
<td>Worst case: $Y = (n_i - i)r$, $X = n_r$</td>
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Table 3.

<table>
<thead>
<tr>
<th>(t)</th>
<th>States (k_i)</th>
<th>(w_i^{(1)})</th>
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<td>-</td>
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<td>-</td>
<td>0</td>
<td>8</td>
<td>0.064</td>
<td>7</td>
<td>0.352</td>
</tr>
</tbody>
</table>

(1) \(w_i\) = Threshold above which outsourcing is optimal ("-" means there is no threshold; i.e., \(a=0\) is optimal for all states in that group). \(q_i\) = Probability that the threshold is crossed.
Table 4.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$k_i$</th>
<th>$w_i^{(1)}$</th>
<th>$q_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2.1</td>
<td>7</td>
<td>0.0002</td>
</tr>
<tr>
<td>4</td>
<td>1.8</td>
<td>7</td>
<td>0.0007</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1.2</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>2.7</td>
<td>8</td>
<td>0.0004</td>
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<tr>
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<td>2.4</td>
<td>8</td>
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<tr>
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<td>-</td>
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</tr>
</tbody>
</table>

(1): $w_i$ = Threshold above which outsourcing is optimal (“-” means there is no threshold; i.e., $a=0$ is optimal for all states in that group). $q_i$ = Probability that the threshold is crossed.
Table 5.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$k_i$</th>
<th>$w_j^{(1)}$</th>
<th>$q_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
<td>3</td>
<td>2.5</td>
<td>1</td>
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<tr>
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<td>2</td>
<td>3</td>
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<td>1.5</td>
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<td>0.875</td>
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<td>0.812</td>
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<tr>
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<td>2</td>
<td>6</td>
<td>0.687</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

(1): $w_j = \text{Threshold above which outsourcing is optimal ("-" means there is no threshold; i.e.,}$

$a=0$ is optimal for all states in that group). $q_i = \text{Probability that the threshold is crossed.}$
Table 6.

<table>
<thead>
<tr>
<th>Case</th>
<th>Conditioning argument</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$m &lt; nr - r$</td>
</tr>
<tr>
<td>2</td>
<td>$nr - r \leq m &lt; nr$</td>
</tr>
<tr>
<td>3</td>
<td>$nr \leq m$</td>
</tr>
</tbody>
</table>
Captions for Figures and Tables:

Figure 1. Relationship between RL chain and costs considered in the MDM.

Figure 2. Criterion followed for a strict partial state ordering.

Table 1. Cases for Condition 3 when $n, r < k_i$

Table 2. Cases for Condition 3 when $k_i \leq (n_i - i)r$

Table 3. Value of the threshold and the probability of crossing it for $r = \{0.2, 0.3, 0.4, 0.5\}$

Table 4. Value of the threshold and the probability of crossing it for Case 1: $r = 0.3, \ L = 5$

Table 5. Value of the threshold and the probability of crossing it for Case 2: $r = 0.5, \ L = 4$

Table 6. Cases considered for Case (a) in Lemma 1.