MATH 517: HOMEWORK 5  
SPRING 2016

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Numerical Methods for ODE IVPs

1. Consider the Leapfrog method

   \[ U^{n+1} = U^{n-1} + 2k f(U^n) \]

   applied to the test problem \( u' = \lambda u \). The method is zero-stable and second order accurate, and hence convergent. If \( \lambda < 0 \) then the true solution is exponentially decaying.

   On the other hand, for \( \lambda < 0 \) and \( k > 0 \) the point \( z = k\lambda \) is never in the region of absolute stability of this method, and hence the numerical solution should be growing exponentially for any nonzero time step. (And yet it converges to a function that is exponentially decaying.)

   Suppose we take \( U^0 = \eta \), use Forward Euler to generate \( U^1 \), and then use the midpoint method for \( n = 2, 3, \ldots \). Work out the exact solution \( U^n \) by solving the linear difference equation and explain how the apparent paradox described above is resolved.

2. (a) Find the general solution of the linear difference equation:

   \[ U^{n+3} + 2U^{n+2} - 4U^{n+1} - 8U^n = 0. \]

   (b) Determine the particular solution with initial data \( U_0 = 4, U_1 = -2, U_2 = 8 \).

   (c) Consider the iteration:

   \[
   \begin{bmatrix}
   U^{n+1} \\
   U^{n+2} \\
   U^{n+3}
   \end{bmatrix} = \begin{bmatrix}
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   8 & 4 & -2
   \end{bmatrix} \begin{bmatrix}
   U^n \\
   U^{n+1} \\
   U^{n+2}
   \end{bmatrix}
   \]

   The matrix appearing here is the companion matrix for the difference equation. If this matrix is called \( A \), then we can determine \( U^n \) from the starting values if we know \( A^n \), the \( n \)th power of \( A \). If \( A = RAR^{-1} \) is the Jordan Canonical form for the matrix, then \( A^n = R\Lambda^n R^{-1} \). Determine the eigenvalues and Jordan Canonical form for this matrix and show how this is related to the general solution found in (a).
3. Write a MATLAB script to plot the region of absolute stability of the 4-stage Runge-Kutta method (see Example 5.13 on Page 126).

4. A general Runge-Kutta method has a Butcher tableau of the following form:

\[
\begin{array}{c|c}
\vec{c} & A \\
\hline
\vec{b}^T & \\
\end{array}
\]

where \(A\) is an \(s \times s\) matrix, \(\vec{b} = [b_1, b_2, \ldots, b_s]\), \(\vec{c} = [c_1, c_2, \ldots, c_s]\).

(a) Show that this method when applied to \(u' = \lambda u\) can be written as

\[
\vec{Y} = \lambda U^n (I - zA)^{-1} \vec{c},
\]

where \(z = k\lambda\), \(\vec{Y} = [Y_1, Y_2, \ldots, Y_s]\), \(I\) is the \(s \times s\) identity matrix, and \(\vec{c} = [1, 1, 1, \ldots, 1]\) is a vector of length \(s\).

(b) Use this result to show that

\[
R(z) = 1 + z\vec{b}^T (I - zA)^{-1} \vec{c}.
\]

(c) For the remainder of this problem we concentrate on the explicit case where \(A\) is a strictly lower triangular matrix:

\[
A = \begin{bmatrix}
0 & a_{21} & 0 \\
a_{31} & a_{32} & 0 & \\
& \ddots & \ddots & \ddots \\
a_{s1} & a_{s2} & \cdots & a_{ss-1} & 0
\end{bmatrix}.
\]

Show that taking \(s\) powers of the matrix \(A\) results in the zero matrix:

\(A^s = 0I\).

**HINT:** Prove this result using the Cayley-Hamilton Theorem, which states that if you compute the characteristic polynomial of a matrix \(A\):

\[
p(\lambda) = \det(A - \lambda I),
\]

then you can replace all the \(\lambda\)'s in the characteristic polynomial by the matrix \(A\) and then

\[
p(A) = 0I.
\]

(d) Use the result from Part (c) to show prove that

\[
(I - zA)^{-1} = I + zA + z^2 A^2 + z^3 A^3 + \cdots + z^{s-1} A^{s-1}.
\]

**HINT:** Taylor expand \(f(z) = (I - zA)^{-1}\) about \(z = 0\).

(e) Use the result from Parts (c) and (d) to prove that for any \(s\)-stage explicit Runge-Kutta method \(R(z)\) is a polynomial of degree of at most \(s\).