Multivariate linear recursions with Markov-dependent coefficients☆

Diana Hay[1], Reza Rastegar[1], Alexander Roitershtein[1,*]

Abstract

We study a linear recursion with random Markov-dependent coefficients. In a "regular variation in, regular variation out" setup we show that its stationary solution has a multivariate regularly varying distribution. This extends results previously established for i.i.d. coefficients.

Keywords: random vector equations, multivariate random recursions, stochastic difference equation, tail asymptotic, heavy tails, multivariate regular variation.
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1. Introduction

Let $\mathbf{Q}_n$ be random d-vectors, M_n random $d \times d$ matrices, and consider the recursion

$$\mathbf{X}_n = \mathbf{Q}_n + M_n \mathbf{X}_{n-1}, \qquad \mathbf{X}_n \in \mathbb{R}^d,\ n \in \mathbb{Z}. \tag{1}$$

This equation has been used to model the progression of real-world systems in discrete time, for example, in queuing theory [1] and financial models [2, 3]. See for instance [4, 5, 6, 7] and references therein for more examples.

Let I denote the $d \times d$ identity matrix and let $\Pi_n = M_0 M_{-1} \cdots M_{-n}$ for $n \geq 0$. It is well known (see for instance [8]) that if the sequence $(\mathbf{Q}_n, M_n)_{n \in \mathbb{Z}}$ is stationary and ergodic, and Assumption 1 stated below is imposed, then for any $\mathbf{X}_0$ series $\mathbf{X}_n$ converges in distribution, as $n \to \infty$, to the random equilibrium

$$\mathbf{X} = \mathbf{Q}_0 + \sum_{k=1}^{\infty} \Pi_{-k+1} \mathbf{Q}_{-k},$$

which is the unique initial value making $(\mathbf{X}_n)_{n \geq 0}$ into a stationary sequence.

The stationary solution $\mathbf{X}$ of the stochastic difference equation (1) has been studied by many authors. Assuming the existence of a certain "critical exponent" for M_n, the distribution tails $P(\mathbf{X} \cdot \mathbf{y} > t)$ and $P(\mathbf{X} \cdot \mathbf{y} < -t)$ for a deterministic vector $\mathbf{y} \in \mathbb{R}^d$ were shown to be regularly varied (in fact, power tailed) in [9] (for $d = 1$ an alternative

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*Corresponding author. E-mail: roiterst@iastate.edu
[1]Department of Mathematics, Iowa State University, Ames, IA 50011, USA

proof is given in [10]). Under different assumptions and for $d = 1$ only, similar results for the tails of $\mathbf{X}$ were obtained in [11, 12]. The multivariate recursion (1) and tails of its stationary solution $\mathbf{X}$ were studied in [13, 14, 15] under conditions similar to those of [9], and in [16, 17] extending the one-dimensional setup of [11, 12]. In all the works mentioned above, it is assumed that $(\mathbf{Q}_n, M_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence, and [16, 17] suppose in addition that the sequences $(\mathbf{Q}_n)_{n\in\mathbb{Z}}$ and $(M_n)_{n\in\mathbb{Z}}$ are mutually independent.

The goal of this paper is to extend the results of [11, 12] to the case where $(Q_n, M_n)_{n\in\mathbb{Z}}$ are induced by a Markov chain. The extension is desirable in many, especially financial, applications, see for instance [18, 19, 20]. We remark that in dimension one the results of [9, 10] (where M_n is dominant in determining the tail behavior of $\mathbf{X}$) and [11, 12] (where $\mathbf{Q}_n$ is dominant) were extended to a Markovian setup in [21, 22] and [23], respectively.

2. The setup

For $\mathbf{Q} \in \mathbb{R}^d$ define $\|\mathbf{Q}\| = \max_{1\le i\le d} |\mathbf{Q}(i)|$ and let $\|M\| = \sup_{\mathbf{Q}\in\mathbb{R}^d, \|\mathbf{Q}\|=1} \|M\mathbf{Q}\|$ denote the corresponding operator norm for a $d \times d$ matrix M. The following condition ensures the existence and the uniqueness of the stationary solution to (1). The condition is also known to be close to necessity (see [24]).

Assumption 1.

(A1) $E\big(\log^+ \|M_0\|\big) < +\infty$ *and* $E\big(\log^+ \|\mathbf{Q}_0\|\big) < +\infty$, *where* $x^+ := \max\{x, 0\}$ *for* $x \in \mathbb{R}$.

(A2) The top Lyapunov exponent $\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|M_1 M_2 \cdots M_n\|$ *is strictly negative.*

Let $\mathbf{I}_A$ denote the indicator function of the set A, that is $\mathbf{I}_A$ is one or zero according to whether the event A occurs or not.

Definition 1. *The coefficients* $(\mathbf{Q}_n, M_n)_{n\in\mathbb{Z}}$ *are said to be induced by a sequence of random variables* $(Z_n)_{n\in\mathbb{Z}}$*, each valued in a finite set* $\mathcal{D}$*, if there exists a sequence of independent random pairs* $(\mathbf{Q}_{n,i}, M_{n,i})_{n\in\mathbb{Z}, i\in\mathcal{D}}$ *with* $\mathbf{Q}_{n,i} \in \mathbb{R}^d$ *and* $M_{n,i}$ *being* $d \times d$ *matrices, such that for a fixed* $i \in \mathcal{D}$, $(\mathbf{Q}_{n,i}, M_{n,i})_{n\in\mathbb{Z}}$ *are i.i.d and*

$$\mathbf{Q}_n = \sum_{j\in\mathcal{D}} \mathbf{Q}_{n,j} \mathbf{I}_{\{Z_n=j\}} = \mathbf{Q}_{n,Z_n} \qquad \text{and} \qquad M_n = \sum_{j\in\mathcal{D}} M_{n,j} \mathbf{I}_{\{Z_n=j\}} = M_{n,Z_n}. \tag{2}$$

Notice that the randomness of the coefficients $(\mathbf{Q}_n)_{n\in\mathbb{Z}}$ induced by a sequence $(Z_n)_{n\in\mathbb{Z}}$ is due to two factors:

1) to the randomness of the underlying auxiliary process $(Z_n)_{n\in\mathbb{Z}}$, which can be thought as representative of the "state of the external world,"

   and, given the value of Z_n,

2) to the "intrinsic" randomness of characteristics of the system which is captured by the random pairs $(\mathbf{Q}_{n,Z_n}, M_{n,Z_n})$.

The independence of $\mathbf{Q}_{n,i}$ and $M_{n,i}$ is not supposed in the above definition. Note that when $(Z_n)_{n\in\mathbb{Z}}$ is a finite Markov chain, (2) defines a *Hidden Markov Model* (HMM). See for instance [25] for a survey of HMM and their applications in various areas.

We will further assume that the vectors $\mathbf{Q}_{n,i}$ are *multivariate regularly varying.* Heavy tailed HMM have been considered for instance in [26], see also references therein. Recall that, for $\alpha \in \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$ is *regularly varying of index* α if $f(t) = t^\alpha L(t)$ for some $L(t) : \mathbb{R} \to \mathbb{R}$ such that $L(\lambda t) \sim L(t)$ for all $\lambda > 0$ (that is $L(t)$ is *slowly varying*). Here and henceforth $f(t) \sim g(t)$ (we will omit "$t \to \infty$" as a rule) means $\lim_{t\to\infty} f(t)/g(t) = 1$.

Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$ with respect to the norm $\|\cdot\|$.

Definition 2. *A random vector* $\mathbf{Q} \in \mathbb{R}^d$ *is said to be regularly varying with index* $\alpha > 0$ *if there exist a function* $\mathbf{a} : \mathbb{R} \to \mathbb{R}$ *regularly varying with index* $1/\alpha$ *and a finite Borel measure* $\mathfrak{S}_{\mathbf{Q}}$ *on* S^{d-1} *such that for all* $t > 0$,

$$nP\big(\|\mathbf{Q}\| > ta_n;\ \mathbf{Q}/\|\mathbf{Q}\| \in \cdot\big) \overset{v}{\to}_{n\to\infty} = t^{-\alpha}\mathfrak{S}_{\mathbf{Q}}(\cdot), \qquad \text{as } n \to \infty, \tag{3}$$

where $\overset{v}{\to}$ *denotes the vague convergence on* S^{d-1} *and* $a_n := \mathbf{a}(n)$.

We denote by $\mathcal{R}_{d,\alpha,\mathbf{a}}$ *the set of all d-vectors regularly varying with index* α, *associated with function* $\mathbf{a}$ *by* (3).

Let E be a locally compact Hausdorff topological space. The vague convergence of measures $\nu_n \overset{v}{\to}_{n\to\infty} \nu$ for finite measures ν_n, $n \geq 0$, and ν on E means (see for instance Proposition 3.12 in [27]) that $\limsup_{n\to\infty} \nu_n(K) \leq \nu(K)$ for all compact $K \subset E$ and $\liminf_{n\to\infty} \nu_n(G) \geq \nu(G)$ for all relatively compact open sets $G \subset E$. In this paper we consider vague convergence on either S^{d-1} or $\overline{\mathbb{R}}_0^d := [-\infty, \infty]^d \backslash \{\mathbf{0}\}$, where $\mathbf{0}$ stands for the zero vector in $\mathbb{R}^d$. In both spaces the topology is inherited from $\mathbb{R}^d$ (in the case of $\overline{\mathbb{R}}_0$ by adding neighborhoods of infinity and removing neighborhoods of zero, see for instance [28] for more details) and can be defined using an appropriate metric making both into a locally compact Polish (complete separable metric) space. A set $K \subset \overline{\mathbb{R}}_0^d$ is relatively compact if its closure does not include $\mathbf{0}$, which makes the space $\overline{\mathbb{R}}_0^d$ especially useful when convergence of regularly varying distributions is considered.

The definition (3) is norm-independent and turns out to be equivalent to the following condition (see for instance [28, 29] or [30]):

> There is a Radon measure ν on $\overline{\mathbb{R}}_0^d$ such that $nP\big(a_n^{-1}\mathbf{Q} \in \cdot\big) \overset{v}{\to}_{n\to\infty} \nu(\cdot)$. The measure ν is referred to as the *measure of regular variation* associated with $(\mathbf{Q}, \mathbf{a})$.

The regular variation of a random vector $\mathbf{Q} \in \mathbb{R}^d$ implies that its one-dimensional projections have regularly varying tails of a similar structure. More precisely, if $\mathbf{Q}$ is regularly varying then for any $\mathbf{x} \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{P\big(\mathbf{Q}\cdot\mathbf{x} > t\big)}{t^{-\alpha}L(t)} = w(\mathbf{x}) \tag{4}$$

for a slowly varying function L and some $w(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ which is not identically zero. The property (4) was used as a definition of regular variation in [9], and it turns out to be equivalent to (3) for all non-integer α as well as for odd integers provided that $\mathbf{Q}$ has non-negative components with a positive probability [31]. The question whether (4) and (3) are equivalent for even integers α in higher dimensions remains open.

In this paper we impose the following conditions on the coefficients $(\mathbf{Q}_n, M_n)_{n\in\mathbb{Z}}$.

Assumption 2. *Let $(Z_n)_{n\in\mathbb{Z}}$ be an irreducible Markov chain with transition matrix H and stationary distribution π defined on a finite state space $\mathcal{D}$. Suppose that the coefficients $(\mathbf{Q}_n, M_n)_{n\in\mathbb{Z}}$ in (1) are induced by the stationary sequence $(Z_n)_{n\in\mathbb{Z}}$, Assumption 1 is satisfied, and, in addition, there exist a constant $\alpha > 0$ and a regularly varying function $\mathbf{a} : \mathbb{R} \to \mathbb{R}$ such that*

(A3) For each $i \in \mathcal{D}$, $\mathbf{Q}_{0,i} \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with an associated measure of regular variation μ_i.

(A4) $\Lambda(\beta) := \limsup_{n\to\infty} \frac{1}{n} \log E\big(\|\Pi_{-n}\|^\beta\big) < 0$ for some $\beta > \alpha$. In particular,

$$\text{There exists } m > 0 \text{ such that } E\big(\|\Pi_{-m}\|^\alpha\big) < 1 \text{ and } E\big(\|\Pi_{-m}\|^\beta\big) < 1. \tag{5}$$

3. Main result

The following theorem extends results of [11, 12, 16, 23] to multivariate recursions of the form (1) with Markov-dependent coefficients.

Theorem 1. *Let Assumptions 2 hold. Then $\mathbf{X} \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with measure of regular variation $\mu_{\mathbf{X}}(\cdot) = \sum_{k=-\infty}^{0} E\big(\mu_{Z_k} \circ \Pi_{k+1}^{-1}(\cdot)\big)$, where $\mu \circ \Pi^{-1}(\cdot)$ stands for $\mu\big(\{\mathbf{x} : \Pi\mathbf{x} \in \cdot\}\big)$.*

The theorem is an instance of the phenomenon "regular variation in, regular variation out" for the model (1). We remark that the mechanisms leading to regularly varying tails of $\mathbf{X}$ are quite different in [12, 11] versus [9, 10]. In the former case, Kesten's "critical exponent" is not available, and therefore more explicit assumptions about distribution of $\mathbf{Q}_n$ are made. Then $\mathbf{Q}_n$ dominates and creates cumulative effects, namely $\mathbf{X}$ turns out to be regularly varying as a sum of regularly varying terms $\Pi_{n+1}\mathbf{Q}_n$. The setup of Assumption 2 is particularly appealing because a similar "cumulative effect" enables one to gain insight into the structure and fine properties of the sequence $(\mathbf{X}_n)_{n\in\mathbb{N}}$, in particular into the asymptotic behavior of both the partial sums as well as multivariate extremes of $(\mathbf{X}_n)_{n\in\mathbb{N}}$, see for instance [6, 32, 33, 34, 35, 36].

The proof of Theorem 1 is deferred to the Appendix which is included at the end of this paper. The proof combines ideas developed in [11], [16], and [23]. We notice that Grey conjectured in [11] that using his method it may be possible to extend the results of [16] and rid of the assumption that $(\mathbf{Q}_n)_{n\in\mathbb{Z}}$ and $(M_n)_{n\in\mathbb{Z}}$ are independent. We accomplish here the program suggested by Grey, and in fact extend it further to coefficients induced by a finite-state irreducible Markov chains.

4. Concluding remarks

The stochastic difference equation (1), in particular regular variation of the distribution tails of its stationary solutions, has been studied by many authors. The equation has a remarkable variety of both theoretical as well as real-world applications. For examples in theoretical probability see for instance [37, 38, 39, 40, 41]. For examples of applications in economics see fee for instance [18, 19, 20, 42, 43]. In this paper we showed that the tails of the stationary solution to the multivariate equation are regularly varying in a Markovian "regular variation in, regular variation out" setup, extending known i.i.d results. The main result of the paper is stated in Theorem 1. The extension to Markovian coefficients seems to be desirable in many, especially financial, applications.

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Appendix. Proof of Theorem 1

The following result extends Lemma 2 in [11] and the relation (2.4) in [16]. Notice, that in contrast to [16] we do not assume that $\mathbf{Q}$ and M are independent.

Lemma 1. *Let* $\mathbf{Y}, \mathbf{Q}$ *be random d-vectors and* Π *be a random* $d \times d$ *matrix such that*

(i) $\mathbf{Q}$ *is independent of the pair* $(\mathbf{Y}, \Pi)$

(ii) For some constant $\alpha > 0$ *and regularly varying* $\mathbf{a} : \mathbb{R} \to \mathbb{R}$, $\mathbf{Y}$ *and* $\mathbf{Q}$ *belong to* $\mathcal{R}_{d,\alpha,\mathbf{a}}$ *with associated measures of regular variation measures* ν *and* μ, *respectively.*

(iii) $E\big(\|\Pi\|^\beta\big) < \infty$ *for some* $\beta > \alpha$.

Then, $\mathbf{Y}+\Pi\mathbf{Q} \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ *with associated measure of regular variation* $\nu(\cdot)+E\big(\mu\circ\Pi^{-1}(\cdot)\big)$.

Proof. We need to show that for any compact set $K \subset S^{d-1}$,

$$\limsup_{n\to\infty} nP\big(\|\mathbf{Y}+\Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in K\big) \le t^{-\alpha}\big[\mathfrak{S}_{\mathbf{Y}}(K) + E\big(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(K\,)\big)\big] \quad (6)$$

while for any open set $G \subset S^{d-1}$,

$$\liminf_{n\to\infty} nP\big(\|\mathbf{Y}+\Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in G\big) \ge t^{-\alpha}\big[\mathfrak{S}_{\mathbf{Y}}(G) + E\big(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(G\,)\big)\big] \quad (7)$$

To this end, we will use a decomposition resembling the one exploited in [11, Lemma 2] and [23, Proposition 2.1]. Namely, we fix $\varepsilon > 0$ and write for any Borel set $A \subset S^{d-1}$, $nP\big(\|\mathbf{Y}+\Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in A\big) = J^{(1)}_{t,A}(n) - J^{(2)}_{t,A}(n) + J^{(3)}_{t,A}(n) + J^{(4)}_{t,A}(n)$, where

$$\begin{aligned}
J^{(1)}_{t,A}(n) &= nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in A\big),\\
J^{(2)}_{t,A}(n) &= nP\big(\|\mathbf{Y}\| > (1+\varepsilon)ta_n,\ \|\mathbf{Y}+\Pi\mathbf{Q}\| \le ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in A\big)\\
J^{(3)}_{t,A}(n) &= nP\big((1-\varepsilon)ta_n < \|\mathbf{Y}\| \le (1+\varepsilon)ta_n, \|\mathbf{Y}+\Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in A\big)\\
J^{(4)}_{t,A}(n) &= nP\big(\|\mathbf{Y}\| \le (1-\varepsilon)ta_n, \|\mathbf{Y}+\Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in A\big).
\end{aligned}$$

Fix a constant $\delta \in (0,1)$ and let $K \subset S^{d-1}$ be an arbitrary compact set. Then $J^{(1)}_{t,K}(n) \le nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}} \in K^{\delta}\big) + nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y}+\Pi\mathbf{Q}}\| > \delta\big)$. It is not hard to check that for any constant $\gamma > 0$ and vectors $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{R}}^d_0$,

$$\|\overline{\mathbf{y}} - \overline{\mathbf{x}+\mathbf{y}}\| > \gamma \text{ implies } \|\mathbf{x}\| > \frac{\gamma\|\mathbf{y}\|}{2+\gamma}. \quad (8)$$

Thus $nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y}+\Pi\mathbf{Q}}\| > \delta\big) \le nP\Big(\|\mathbf{Y}\| > ta_n, \|\Pi\|\|\mathbf{Q}\| > \frac{\delta ta_n}{3}\Big) \le nP\big(\|\mathbf{Y}\| > ta_n\big)P\Big(\|\mathbf{Q}\| \ge \frac{\delta ta_n^{\frac{\beta-\alpha}{2\beta}}}{3}\Big) + nP\big(\|\Pi\| \ge a_n^{\frac{\alpha+\beta}{2\beta}}\big)$. Since $P\big(\|\Pi\| \ge a_n^{\frac{\alpha+\beta}{2\beta}}\big) \le a_n^{-\frac{\alpha+\beta}{2}} E\big(\|\Pi\|^{\beta}\big)$, we have $\limsup_{n\to\infty} nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y}+\Pi\mathbf{Q}}\| > \delta\big) = 0$. Thus

$$\limsup_{n\to\infty} J^{(1)}_{t,K}(n) \le \lim_{\delta\to 0}\limsup_{n\to\infty} nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}} \in K^{\delta}\big) = t^{-\alpha}\mathfrak{S}_{\mathbf{Y}}(K). \quad (9)$$

Since $J^{(2)}_{t,K}(n) \le nP\big(\|\Pi\| \ge ta_n^{\frac{\alpha+\beta}{2\beta}}\big) + nP\big(\|\mathbf{Y}\| > (1+\varepsilon)ta_n\big)P\big(\|\mathbf{Q}\| \ge \varepsilon ta_n^{\frac{\beta-\alpha}{2\beta}}\big)$, we have

$$\limsup_{n\to\infty} J^{(2)}_{t,K}(n) = 0. \quad (10)$$

Next, $J^{(3)}_{t,K}(n) \le nP\big((1-\varepsilon)ta_n < \|\mathbf{Y}\| \le (1+\varepsilon)ta_n\big) \sim t^{-\alpha}\big[(1-\varepsilon)^{-\alpha} - (1+\varepsilon)^{-\alpha}\big]$. Hence

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} J^{(3)}_{t,K}(n) = 0. \quad (11)$$

Define $g_n(\mathbf{x}, A) = nP\big(\|\mathbf{Y}+\Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y}+\Pi\mathbf{Q}} \in K \big| \mathbf{Y} = \mathbf{x}, \Pi = A\big)$. Fix constants $\rho > 0$ and $\eta > 0$, and let $J^{(4)}_{t,K}(n) = J^{(4,1)}_{t,K}(n) + J^{(4,2)}_{t,K}(n) + J^{(4,3)}_{t,K}(n)$, where

$$\begin{aligned}
J^{(4,1)}_{t,K}(n) &= E\big(g(\mathbf{Y},\Pi)\mathbf{I}_{\{\|\mathbf{Y}\| \le (1-\varepsilon)ta_n\}}\mathbf{I}_{\{\|\Pi\| > \rho\}}\big)\\
J^{(4,2)}_{t,K}(n) &= E\big(g(\mathbf{Y},\Pi)\mathbf{I}_{\{\|\mathbf{Y}\| \le (1-\varepsilon)ta_n\}}\mathbf{I}_{\{\|\Pi\| \le \rho\}}\mathbf{I}_{\{\|\mathbf{Y}\| > \eta\}}\big)\\
J^{(4,3)}_{t,K}(n) &= E\big(g(\mathbf{Y},\Pi)\mathbf{I}_{\{\|\mathbf{Y}\| \le (1-\varepsilon)ta_n\}}\mathbf{I}_{\{\|\Pi\| \le \rho\}}\mathbf{I}_{\{\|\mathbf{Y}\| \le \eta\}}\big).
\end{aligned}$$

The first two terms tend to zero as η and ρ go to infinity. More precisely,

$$\begin{aligned}
\limsup_{n\to\infty} J_{t,K}^{(4,1)}(n) &\leq \limsup_{n\to\infty} E\Big(nP\big(\|\Pi\|\cdot\|\mathbf{Q}\| > \varepsilon t a_n \big|\Pi\big)\mathbf{I}_{\{\|\Pi\|>\rho\}}\Big) \\
&= (\varepsilon t)^{-\alpha} E\big(\|\Pi\|^\alpha \mathbf{I}_{\{\|\Pi\|>\rho\}}\big) \to_{\rho\to\infty} 0, \qquad (12)\\
\limsup_{n\to\infty} J_{t,K}^{(4,2)}(n) &\leq \limsup_{n\to\infty} E\big(nP\big(\|\Pi\|\cdot\|\mathbf{Q}\| > \varepsilon t a_n \big|\Pi\big)\mathbf{I}_{\{\|\Pi\|\leq\rho\}}\mathbf{I}_{\{\|\mathbf{Y}\|>\eta\}}\big) \\
&\leq \rho^\alpha (\varepsilon t)^{-\alpha} P\big(\|\mathbf{Y}\|>\eta\big) \to_{\eta\to\infty} 0. \qquad (13)
\end{aligned}$$

To show the asymptotic of $J_{t,K}^{(4,3)}(n)$ as n goes to infinity write,

$$\begin{aligned}
J_{t,K}^{(4,3)}(n) &\leq nP\big(\eta + \|\Pi\mathbf{Q}\| > ta_n, \overline{\Pi\mathbf{Q}} \in K^\delta\big) \\
&+ nP\big(\overline{\mathbf{Y}+\Pi\mathbf{Q}} - \overline{\Pi\mathbf{Q}}\| > \delta, \|\Pi\mathbf{Q}\| \geq \varepsilon t a_n, \|\mathbf{Y}\| \leq \eta\big). \qquad (14)
\end{aligned}$$

Applying the multivariate Breiman's lemma (see for instance [44, Proposition 5.1]) to the first term in the right-hand side of the last inequality and (8) to the second, we obtain $\limsup_{n\to\infty} J_{t,K}^{(4,3)}(n) \leq t^{-\alpha} E\big(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(K)\big)$. Thus (6) is implied by (9)-(14).

It remains to show that (7) holds for any open set $G \subset S^{d-1}$. According to (10), $\limsup_{n\to\infty} J_{t,G}^{(2)}(n) \leq \limsup_{n\to\infty} J_{t,\overline{G}}^{(2)}(n) = 0$. Let $G_k \subset S^{d-1}$, $k \in \mathbb{N}$ be open sets such that $G_k \subset \overline{G}_k \subset G_{k+1} \subset G$ Let $\gamma_k = \frac{1}{2}\inf\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x} \in G_k, \mathbf{y} \in G^c\}$. Then, $J_{t,G}^{(1)}(n) \geq nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}} \in G_k\big) - nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y}+\Pi\mathbf{Q}}\| > \gamma_k\big)$. By (8), $\liminf_{n\to\infty} J_{t,G}^{(1)}(n) \geq \lim_{\varepsilon\to 0}\liminf_{n\to\infty} nP\big(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}} \in G_k\big) = t^{-\alpha}\mathfrak{S}_{\mathbf{Y}}(G_k)$. Letting $k \to \infty$ we obtain $\liminf_{n\to\infty} J_{t,G}^{(1)}(n) \geq t^{-\alpha}\mathfrak{S}_{\mathbf{Y}}(G)$. To conclude, observe that

$$\begin{aligned}
J_{t,G}^{(4,3)}(n) &\geq nP\big(\|\Pi\mathbf{Q}\| - \eta > ta_n, \overline{\Pi\mathbf{Q}} \in G_k\big) - nP\big(\rho\|\mathbf{Q}\| - \eta > ta_n; \|\mathbf{Q}\| \leq \eta\big) \\
&- nP\big(\|\overline{\mathbf{Y}+\Pi\mathbf{Q}} - \overline{\Pi\mathbf{Q}}\| > \gamma_k, \|\Pi\mathbf{Q}\| \geq \varepsilon t a_n, \|\mathbf{Y}\| \leq \eta\big).
\end{aligned}$$

By (8), $\liminf_{n\to\infty} J_{t,G}^{(4,3)}(n) \geq t^{-\alpha} E\big(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(G_k)\big)$. Letting $k \to \infty$ establishes (7). □

The next lemma, which generalizes Proposition 2.1 of [23], is the key element of our proof of Theorem 1.

Lemma 2. *Let Assumption 2 hold. Fix an integer $k \leq -1$ and let $\mathbf{Y}_{k+1} \in \mathbb{R}^d$ be a random vector such that $\mathbf{Y}_{k+1} \in \sigma(Z_n, \mathbf{Q}_n, M_n : n \geq k+1)$. Let $\mathbf{Y}_k = \mathbf{Y}_{k+1} + \Pi_{k+1}\mathbf{Q}_k$ and write $\mathbf{Y}_k = \sum_{i\in\mathcal{D}} \mathbf{Y}_{k,i}\mathbf{I}_{\{Z_k=i\}}$.*

Then, each vector $\mathbf{Y}_{k,i}$ belongs to $\mathcal{R}_{d,\alpha,\mathbf{a}}$ with associated measure of regular variation $\nu_{k,i} := E\big(\nu_{k+1,Z_{k+1}}(\cdot)\mathbf{I}_{\{Z_k=i\}}\big) + E\big(\mu_i \circ \Pi_{k+1}^{-1}(\cdot)\mathbf{I}_{\{Z_k=i\}}\big)$, and hence $\mathbf{Y}_k \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with associated measure of regular variation $E\big(\nu_{k+1,Z_{k+1}}(\cdot)\big) + E\big(\mu_{Z_k} \circ \Pi_{k+1}^{-1}(\cdot)\big)$.

Proof. Since $P\big((\mathbf{Q}_k, \Pi_{k+1}, \mathbf{Y}_{k+1}) \in \cdot | Z_{k+1} = i, Z_k = j\big) = P\big((\mathbf{Q}_{1,j}, \Pi_{k+1,i}, \mathbf{Y}_{k+1,i}) \in \cdot\big)$, using Lemma 1 we obtain for Borel subsets $A \subset \overline{\mathbb{R}}_0^d$,

$$\begin{aligned}
&\lim_{n\to\infty} P\big(a_n^{-1}\mathbf{Y}_{k,i} \in A\big) \\
&\quad = \lim_{n\to\infty} \sum_{j\in\mathcal{D}} P\big(\mathbf{Y}_{k+1} + \Pi_{k+1}\mathbf{Q}_k \in a_n A \big| Z_{k+1} = j, Z_k = i\big)\pi_i H(i,j) \\
&\quad = \lim_{n\to\infty} \sum_{j\in\mathcal{D}} P\big(\mathbf{Y}_{k+1,j} + \Pi_{k+1,j}\mathbf{Q}_{k,i} \in a_n A\big)\pi_i H(i,j). \\
&\quad = \sum_{j\in\mathcal{D}} \big[\nu_{k+1,j}(A) + E\big(\mu_i \circ \Pi_{k+1,j}^{-1}(A)\big)\big]\pi_i H(i,j) \\
&\quad = E\big(\nu_{k+1,Z_{k+1}}(A)\mathbf{I}_{\{Z_k=i\}}\big) + E\big(\mu_i \circ \Pi_{k+1}^{-1}(A)\mathbf{I}_{\{Z_k=i\}}\big).
\end{aligned}$$

The proof of the lemma is completed. □

We are now in position to complete the proof of Theorem 1. First we introduce some notations. Throughout the rest of the paper:

For a constant $\delta > 0$ and a set K (either in S^{d-1} or $\overline{\mathbb{R}}_0^d$), let K^δ denote the closed δ-neighborhood of K, that is $K^\delta = \{\mathbf{x} : \exists\, \mathbf{y} \in K \text{ s.t. } \|\mathbf{x}-\mathbf{y}\| \le \delta\}$. For $\mathbf{x} \in \mathbb{R}^d/\{0\}$, let $\overline{\mathbf{x}}$ denote its direction $\mathbf{x}/\|\mathbf{x}\|$. For a set G, let $\overline{G}$ denote its closure $\bigcap_{\delta>0} G^\delta$.

The final step in the proof is similar to the corresponding argument in [16], and is reproduced here for the sake of completeness. It follows from Lemma 2 that, for any $L \in \mathbb{N}$ and Borel $A \subset \overline{\mathbb{R}}_0^d$,

$$\lim_{n\to\infty} nP\Big(\sum_{k=-L}^{0} \Pi_{k+1}\mathbf{Q}_k \in a_n A\Big) = \sum_{k=-L}^{0} E\big(\mu_{Z_k} \circ \Pi_{k+1}^{-1}(A)\big), \tag{15}$$

while [23, Theorem 1.4] yields with the help of (5) that for any constant $\delta > 0$,

$$\lim_{L\to\infty} \limsup_{n\to\infty} nP\Big(\sum_{k=-\infty}^{-L-1} \|\Pi_{k+1}\| \cdot \|\mathbf{Q}_k\| > \delta a_n\Big) = 0. \tag{16}$$

For a compact set $K \subset \overline{\mathbb{R}}_0^d$, we have
$P\Big(\sum\limits_{k=-\infty}^{0} \Pi_{k+1}\mathbf{Q}_k \in a_n K\Big) \le P\Big(\sum\limits_{k=-L}^{0} \mathbf{Q}_k\Pi_{k+1} \in a_n K^\delta\Big) + P\Big(\sum\limits_{k=-\infty}^{-L-1} \|\Pi_{k+1}\mathbf{Q}_k\| > \delta a_n\Big)$.
Hence, $\limsup_{n\to\infty} nP\Big(\frac{1}{a_n}\sum_{k=-\infty}^{0} \Pi_{k+1}\mathbf{Q}_k \in K\Big) \le \sum_{k=-L}^{0} E\big(\mu_{Z_k} \circ \Pi_{k+1}^{-1}(K^\delta)\big)$ in virtue of (15) and (16). Letting then $\delta \to 0$, we obtain

$$\limsup_{n\to\infty} nP\Big(a_n^{-1}\sum_{k=-\infty}^{0} \Pi_{k+1}\mathbf{Q}_k \in K\Big) \le \sum_{k=-\infty}^{0} E\big(\mu_{Z_k} \circ \Pi_{k+1}^{-1}(K)\big). \tag{17}$$

Let $G \subset \overline{\mathbb{R}}_0^d$ be relatively compact and open. Consider open relatively compact sets $G_k \subset \overline{\mathbb{R}}_0^d$, $k \in \mathbb{N}$, such that $G_k \subset \overline{G}_k \subset G_{k+1} \subset G$. For any m, L, there is $\varepsilon > 0$ such that

$\Big\{\sum_{k=-L}^{0}\Pi_{k+1}\mathbf{Q}_k\in a_nG_m\Big\}\bigcup\Big\{\Big\|\sum_{k=-\infty}^{-L-1}\Pi_{k+1}\mathbf{Q}_k\Big\|\le\varepsilon a_n\Big\}\subset\Big\{\sum_{k=-\infty}^{0}\Pi_{k+1}\mathbf{Q}_k\in a_nG\Big\}$.
Therefore, with $\mathcal{F}_0:=\sigma(M_n,Z_n:n\le 0)$, we have for any G_m,

$$\begin{aligned}\liminf_{n\to\infty} nP\Big(a_n^{-1}\sum_{k=-\infty}^{0}\Pi_{k+1}\mathbf{Q}_k\in G\Big)&=\liminf_{n\to\infty} nE\Big[P\Big(a_n^{-1}\sum_{k=-\infty}^{0}\Pi_{k+1}\mathbf{Q}_k\in G|\mathcal{F}_0\Big)\Big]\\ =\liminf_{n\to\infty}E\Big[nP\Big(a_n^{-1}&\sum_{k=-L}^{0}\Pi_{k+1}\mathbf{Q}_k\in G_m\Big|\mathcal{F}_0\Big)P\Big(\Big\|\sum_{k=-\infty}^{-L-1}\Pi_{k+1}\mathbf{Q}_k\Big\|\le\varepsilon a_n\Big|\mathcal{F}_0\Big)\Big]\\ \ge E\Big[\liminf_{n\to\infty}nP\Big(a_n^{-1}&\sum_{k=-L}^{0}\Pi_{k+1}\mathbf{Q}_k\in G_m\Big|\mathcal{F}_0\Big)P\Big(\Big\|\sum_{k=-\infty}^{-L-1}\Pi_{k+1}\mathbf{Q}_k\Big\|\le a_n\varepsilon\Big|\mathcal{F}_0\Big)\Big],\end{aligned}$$

where for the last inequality we used Fatou's lemma. Hence, (15) yields the lower bound $\liminf_{n\to\infty} nP\Big(a_n^{-1}\sum_{k=-\infty}^{0}\Pi_{k+1}\mathbf{Q}_k\in G\Big)\ge\sum_{k=-L}^{0}E\big(\mu_{Z_k}\circ\Pi_{k+1}^{-1}(G_m)\big)$. Letting $m\to\infty$ and then $L\to\infty$, $\liminf_{n\to\infty} nP\Big(a_n^{-1}\sum_{k=-\infty}^{0}\Pi_{k+1}\mathbf{Q}_k\in G\Big)\ge\sum_{k=-\infty}^{0}E\big(\mu_{Z_k}\circ\Pi_{k+1}^{-1}(G)\big)$. This bound along with (17) yield the claim of the theorem provided that we have shown that $\mu_{\mathbf{X}}(\cdot)=E\big(\mu_{Z_k}\circ\Pi_{k+1}^{-1}(\cdot)\big)$ is a Radon measure on $\overline{\mathbb{R}}_0^d$, that is (see for instance Remark 3.3 in [17]) $\mu_{\mathbf{X}}(K)<\infty$ for any compact set $K\in\overline{\mathbb{R}}_0^d$. Toward this end notice that since $\epsilon_K:=\inf_{\mathbf{x}\in K}\|\mathbf{x}\|>0$ and in virtue of *(A4)* of Assumption 2,

$$\begin{aligned}\mu_{\mathbf{X}}(K)\le\sum_{k=-\infty}^{0}E\Big[\sum_{i\in\mathcal{D}}\mu_i\circ\Pi_{k+1}^{-1}(K)\Big]&=\sum_{k=-\infty}^{0}E\Big[\sum_{i\in\mathcal{D}}\mu_i\big(\{\mathbf{x}:\Pi_{k+1}\mathbf{x}\in K\}\big)\Big]\\ \le\sum_{k=-\infty}^{0}E\Big[\sum_{i\in\mathcal{D}}\mu_i\big(\{\mathbf{x}:\|\mathbf{x}\|\ge\epsilon_K\|\Pi_{k+1}\|^{-1}\}\big)\Big]&=\sum_{k=-\infty}^{0}|\mathcal{D}|\epsilon_K^{-\alpha}\,E\big(\|\Pi_{k+1}\|^{\alpha}\big)<\infty,\end{aligned}$$

completing the proof of the theorem. □