Math 267, Ordinary differential equations

Christian Roettger

382 Carver Hall
Mathematics Department
Iowa State University
www.iastate.edu/~roettger

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Chapter 3: Power series

Power Series

3.1 Examples, Operations on P.S.
3.2 Series solutions near ordinary points
First examples I

Definition
A series of the form
\[ \sum_{n=0}^{\infty} c_n (x - a)^n \]
is called a power series centered at \( a \), with coefficients \( c_n \).

Example
You will have seen the geometric series
\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}. \]
First examples II

Other series which you should know are the power series for $\exp x$, $\cos x$, $\sin x$, and $\ln(1 + x)$.

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$$

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$
First examples III

Definition
If for all $x$ in an open interval $(a - r, a + r)$

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$$

then we say that the function $f$ is represented by this power series. We say that this series is centered at $a$, has coefficients $c_n$, and if $r$ is chosen as big as possible then we say $r$ is its radius of convergence. If a function $f$ is represented by a power series in some open interval $(a - r, a + r)$ then we say that $f$ is analytic at $a$.

Example All the examples above are series centered at 0. The coefficients of the series for $\cos x$ are $1, 0, -1/2, 0, 1/4!, 0, -1/6!, 0, 1/8!, \ldots$ (every coefficient of an odd
power of $x$ is zero). Get a feel for the other examples by writing down the first few coefficients!
The geometric series and the series for $\ln(1 + x)$ have convergence radius $r = 1$, the other examples above have convergence radius $\infty$. The function $f(x) = \sin(x)/x$ is defined at $x = 0$. But if we define it by $f(0) = 1$ then it IS analytic - you can take the power series of the sine and divide that by $x$ termwise.
Example 1’ Solve the IVP consisting of the DiffEq

\[ y' + 2y = 6 \]  \hspace{1cm} (1)

and the initial condition \( y(0) = 1 \).
Using power series for DiffEqs II

Figure: Solution of $y' + 2y = 6$ with $y(0) = 1$ and 7 Taylor polynomials.
Solution Assume the solution can be represented by a power series $y(x) = \sum c_n x^n$. Then

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n + 1) c_{n+1} x^n. \quad (2)$$

This relies on theorem 1 below. Note how the subscript $n = 0$ can be omitted from the first sum (why?) and how $n$ is replaced by $n + 1$ in the second sum, so we can read off the coefficient of $x^n$. This also means the limits of the sum are from $n = 0$ to $\infty$ again. Substitute (2) into the DiffEq. (1) to get

$$\sum_{n=0}^{\infty} [(n + 1) c_{n+1} + 2 c_n] x^n = 6.$$
Using power series for DiffEqs IV

We can compare coefficients of like powers just as we did for polynomials! this relies on Theorem 2 below. So we get equations

\[ c_1 + 2c_0 = 6 \]  \hspace{1cm} (3)

\[ c_{n+1} = -\frac{2c_n}{n+1} \quad \text{for } n \geq 1 \]  \hspace{1cm} (4)

The initial condition \( y(0) = 1 \) gives \( c_0 = 1 \) and then Equation (3) gives \( c_1 = 4 \). Thereafter, we can apply (4) and get

\[ c_2 = -\frac{2 \cdot 4}{2} = -4, \]

\[ c_3 = \frac{8}{3}, \ldots \]

\[ c_n = \frac{(-1)^{n-1}2^{n-1}c_1}{n!} \quad \text{for } n \geq 1 \]  \hspace{1cm} (5)
Using power series for DiffEqs V

Substitute (5) into the original power series and we discover an exponential by comparing with the list of example series! Since

\[ \exp(-2x) = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{n!}, \]

we can multiply that series by \(-c_1/2\) to get our series for \(y(x)\) apart from the first coefficient. Altogether,

\[ y(x) = -\frac{c_1}{2} (\exp(-2x) - 1) + c_0 = -2 \exp(-2x) + 3 \]

which is the red curve plotted in the graphic above.

Note that methods from Chapter 2 (characteristic equation \(r + 2 = 0\) therefore general solution \(y_c(x) = e^{-2x}\), particular solution \(...\)) give the same solution and would indeed be faster in this example.
Why power series? I

Power series have two advantages:

- we can also use them for higher-order equations with non-constant coefficients
- we can approximate the solution by the polynomials

\[ T_m(x) = \sum_{n=0}^{m} c_n(x - a)^n. \] (6)

Definition
The polynomial in (6) is called *Taylor polynomial for y(x) of order m*.

Several of these are plotted together with y(x) in the figure above. Taylor polynomials are excellent approximations for x 'close to a', but useless 'far away from a' - and there is no hard rule on how close is 'close'.
Theorems ... I

We can differentiate power series just like polynomials.

**Theorem (1 - Termwise diff.)**

*If a power series* \( y(x) = \sum c_n(x - a)^n \) *converges for* \( x \) *in* \( (a - r, a + r) \) *then* \( y \) *is differentiable and*

\[
y'(x) = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1} = \sum_{n=0}^{\infty} (n + 1)c_{n+1}(x - a)^n
\]

*(these sums also converge for all* \( x \) *in* \( (a - r, a + r) \)).
Theorem (2 - Termwise agreement, identity principle)

If two power series represent the same function on an open interval,

\[ \sum_{n=0}^{\infty} c_n(x - a)^n = y(x) = \sum_{n=0}^{\infty} d_n(x - a)^n \]

then the two series agree coefficientwise, \( c_n = d_n \) for all \( n \geq 0 \).

The next theorem allows (in principle) to compute all coefficients of the power series of a given analytic function.
Theorem (Taylor series)

If the function \( f(x) \) is represented by a power series on an open interval centered at \( a \), then

\[
c_n = \frac{1}{n!} f^{(n)}(a)
\]

where \( f^{(n)}(a) \) means the \( n \)-th derivative of \( f \) at \( a \).

Example    Practice by computing a few coefficients of the Taylor series for \( \tan x \). Then compare to Problem 26.
Example with non-constant coefficients I

Example 2’ Find the general solution of

\[(x - 2)y' + 5y = 0.\]

Solution As in the previous example, start with \(y(x) = \sum_{n=0}^{\infty} c_n x^n\) and use Theorem 1. To find the coefficient of \(x^n\), expand the sum we get for \((x - 2)y'\):

\[
\sum_{n=0}^{\infty} n c_n x^n - 2 \sum_{n=0}^{\infty} (n + 1) c_{n+1} x^n + 5 \sum_{n=0}^{\infty} c_n x^n = 0.
\]

Then use Theorem 2 to compare coefficients,

\[(n + 5)c_n - 2(n + 1)c_{n+1} = 0\]
Example with non-constant coefficients II

for all $n \geq 0$. Apply this repeatedly to express all coefficients in terms of $c_0$,

$$c_n = \frac{(n + 4)!c_0}{2^n n! 4!} = \frac{(n + 1)(n + 2)(n + 3)(n + 4)c_0}{2^n 4!}.$$  \hspace{1cm} (7)

This is none of the example power series in the list above. It can be seen as a constant multiple of the fourth derivative of the geometric series (substitute $x$ by $x/2$). We will be content with solving this equation using the method from Chapter 1, with an integrating factor

$$\rho(x) = \exp \int \frac{5}{x - 2} \, dx = |x - 2|^5$$

and therefore

$$y(x) = C(x - 2)^{-5}.$$
You can compute the Taylor series of $y(x)$ using the Taylor-series theorem and discover that the coefficients are indeed what we found in (7). Note that the series in this example can only converge for $|x - 2| < 2$ - and indeed 2 is the radius of convergence!
Example with non-constant coefficients IV

Figure: Plot of $-32(x - 2)^{-5}$ together with 5 Taylor polynomials
Radius of convergence, R.o.c. I

Theorem (Ratio test for R.o.c.)

If $c_n$ is a sequence of coefficients such that the limit

$$r = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

exists, then $r$ is the radius of convergence of the series

$$\sum c_n(x - a)^n.$$

Example

Practice this with all series we have seen so far! Easy - geometric, exponential, $(x - 2)^{-5}$. Harder - sine and cosine. It helps to rewrite these series in powers of $u = x^2$. For other series which have eg only every third coefficient nonzero, it is helpful to use $u = x^3$. 
Second-order equations I

Example  Find the general solution of

\[ y'' + y = 0. \]

(8)

Solution  Apply Theorem 1 twice to find

\[ y'' = \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n. \]

Substitute into (8) to find the recurrence

\[ c_{n+2} = \frac{-c_n}{(n + 2)(n + 1)}. \]
Second-order equations II

This gives for even/odd $n$

$$
c_{2n} = \frac{(-1)^nc_0}{(2n)!} \quad \text{for } n \text{ even}
$$

$$
c_{2n+1} = \frac{(-1)^{n-1}c_1}{(2n+1)!} \quad \text{for } n \text{ odd}
$$

If $c_0 = 1$ and $c_1 = 0$ then we rediscover the cosine! For $c_0 = 0$ and $c_1 = 1$ we get the sine. The general solution turns out to be

$$
y(x) = c_0 \cos x + c_1 \sin x
$$

which we have already known since Chapter 2.
A nasty surprise I

Example (3’)
Find a solution to $xy' - y = 0$ in powers of $x$. This leads to the recurrence $(n - 1)c_n = 0$ for $n \geq 1$ and $c_0 = 0$. So $c_n = 0$ for all $n \geq 2$ but $c_1$ is arbitrary, $y(x) = c_1 x$ is the general solution. This is a power series, and it has convergence radius $\infty$ but the ratio test fails!!

Example (3”)
Try to solve the initial value problem $xy' + y = 0$, $y(1) = 3$ using a series in powers of $x$. This leads to the recurrence $(n + 1)c_n = 0$ for $n \geq 1$ and $c_0 = 0$. The only solution is the constant function $y(x) = 0$. But the other solutions $y(x) = C/x$ are missed because they cannot be represented by a power series centered at 0.
Ordinary vs. singular points I

Definition
If in a linear differential equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]  \hspace{1cm} (10)

the functions \( p_{n-1}, \ldots, p_0, f \) are all analytic in \( x = a \) then \( a \) is called an ordinary point of the equation, o/w a singular point.

Example
Sometimes you have to rewrite the equation in normal form first, eg

\[ (x - 4)xy'' + \sin(x)y' + 3x^2y = 0 \]

should be divided by \((x - 4)x\) first. Since \(3x/(x - 4)\) and \(\sin(x)/[(x - 4)x]\) are analytic at \(x = 0\), 0 is an ordinary point. But \(x = 4\) is a singular point (the only one).
A much easier way to find the r.o.c. I

Theorem (Distance test for r.o.c.)

If \( x = a \) is an ordinary point for the \( n \)-th order linear equation (10), then there exist \( n \) linearly independent solutions which are also analytic in \( x = a \). The radius of convergence of a power series solution in powers of \( x - a \) is at least equal to the distance from \( a \) to the nearest singularity of any of \( p_{n-1}, \ldots, p_0, f \) in the complex plane.
Example (3“)
Find the general solution $C/x$ of $xy' + y = 0$ in powers of $(x - 2)$. To apply the Distance test theorem, rewrite this as $y' + y/x = 0$. Clearly, the point $x = 0$ is a singular point. So a solution in powers of $(x - 2)$ will have radius of convergence 2 (unless you choose $y = 0$). For practice, find the recurrence relation and do the ratio test, too. Practice with all other series which we have studied!

Example (5’)
Find the general solution in powers of $x$ of

$$(x^2 + 4)y'' + y = 0. \quad (11)$$
Distance test - examples II

Solution The usual approach gives a recurrence

\[ c_{n+2} = -\frac{n^2 - n + 1}{4(n+1)(n+2)} c_n \]

You can use the ratio test, splitting up the series into odd and even powers and using \( u = x^2 \). This gives a r.o.c. of 4 for \( u \) so radius 2 for \( x \). Using the distance test: the singular points for this equation are \( \pm 2i \). The distance from 0 to \( 2i \) is 2. Had we tried a solution in powers of \( x - 5 \), the r.o.c. = distance from 5 to \( 2i \) would have been the distance between points \((5,0)\) and \((0,2)\) in the plane, namely \( \sqrt{2^2 + 5^2} = \sqrt{29} \).

Moral Nice formulas for \( c_n \) are not guaranteed. Make up your mind how many of the \( c_n \) you want to compute, use the recurrence equations and be happy you have a Taylor polynomial as an approximation. Never use the Taylor polynomial for long-range predictions.
Harder Problems I

The Power series method can handle coefficient functions $p_i(x)$ which are not polynomials - the computations just become very involved. If you don’t expect a formula for $c_n$, just want to compute a fixed number of them, this is not too bad.

Example

Solving $xy'' + x^2 y' + \sin(x)y = 0$ with a series in powers of $x$ is possible, and it will even have r.o.c. $\infty$ according to the Distance Test. Part of the process is to express $\sin(x)y$ as a power series. If $y(x) = \sum c_n x^n$, find the coefficient $d_n$ in the series for $\sin(x)y$ up to $n = 4$,

$$\sin(x)y = \sum_{n=0}^{\infty} d_n x^n$$  \hspace{1cm} (12)
Harder Problems II

Solution We begin by noting that

\[ \sin x \approx x - \frac{1}{3} x^3 \]

where we may neglect any higher-order terms because they cannot contribute to the coefficient of \( x^4 \). Similarly, neglect any powers \( x^5 \) or higher in the series for \( y \) and do

\[ \sin(x)y \approx \left( x - \frac{1}{3} x^3 \right) \left( c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \right) \]

\[ = c_0 x + c_1 x^2 + (c_2 - c_0/3)x^3 + (c_3 - c_2/3)x^4 \]

where the \( \approx \)-sign means we neglected all powers of \( x \) of exponent greater than 4. This would only be one step in the process of determining coefficients \( c_n \) of \( y \), we still would have to substitute our result into the equation \( xy'' + x^2 y' + \sin(x)y = 0 \).
Still harder problems would be nonlinear equations like the 'true' pendulum equation

\[ y'' + k \sin y = 0 \]

- even that can be handled by power series as long as you don’t expect nice formulas. Compare also Problem 26 about \( y' + y^2 = 1 \).