Chapter 5 - Linear Transformations

5.1 Introduction

Revise domain, codomain, range, image, pre-image.

**Definition.** Let \( f : V \rightarrow W \) be a function such that

\[
\begin{align*}
(1) \quad & f(v_1 + v_2) = f(v_1) + f(v_2), \\
(2) \quad & f(cv_1) = cf(v_1)
\end{align*}
\]

Then \( f \) is called a linear transformation from \( V \) to \( W \).

Examples - matrices, polynomials, \( \mathbb{R}^n \rightarrow \mathbb{R}^m \).

**Definition.** A linear operator is a linear transformation from \( V \) to \( V \).
Examples - matrices, polynomials, $\mathbb{R}^n \rightarrow \mathbb{R}^n$, geometry.

**Theorem 1**

(1) $L(0) = 0$
(2) $L(-v) = -L(v)$
(3) $L(a_1v_1 + \cdots + a_kv_k) = a_1L(v_1) + \cdots + a_kL(v_k)$

**Theorem 2** The composition of two linear transformation is again a linear transformation.

**Theorem 3** Let $L: V \rightarrow W$ be a linear transformation. The image of any subspace of $V$ under $L$ is a subspace of $W$. In particular, the range $L(V)$ is a subspace of $W$. The pre-image of any subspace of $W$ is a subspace of $V$. In particular, $L^{-1}(0)$ is a subspace of $V$ called the kernel of $L$.

Warning - the symbol $L^{-1}$ does not stand for a linear transformation, only $L^{-1}(S)$ makes sense.
Example. Triangular matrices, symmetric matrices. Problem 4.5.11 from homework has the kernel of $p(x) \mapsto p(2)$. 
5.2 Matrix of a linear transformation

**Theorem 4** Let $B = \{v_1, \ldots v_n\}$ a basis for $V$ and $w_1, \ldots w_n$ be arbitrary vectors in $W$. Then there exists a unique linear transformation $L$ such that for all $i = 1, \ldots, n$

$$L(v_i) = w_i.$$

**Theorem 5** Let $V, W$ be nonzero, finite-dimensional vector spaces. Suppose $B = \{v_1, \ldots v_n\}$ is a basis for $V$, $C$ is a basis for $W$. For any linear transformation $L : V \to W$, there exists a unique $m \times n$-matrix $A_{BC}$ such that

$$[L(v)]_C = A_{BC}[v]_B.$$

Furthermore, the $i$-th column of $A_{BC}$ is $[L(v_i)]_C$. 

\[
\begin{array}{ccc}
V & \xrightarrow{L} & W \\
\downarrow{[\cdot]_B} & & \downarrow{[\cdot]_C} \\
\mathbb{R}^n & \xrightarrow{A_{BC}} & \mathbb{R}^m \\
\end{array}
\]
The matrix $A_{BC}$ is called THE matrix of $L$ wrt $B, C$.

**Theorem 6** Let $L : V \to W$ be a linear transformation with matrix $A_{BC}$ wrt bases $B, C$. Suppose $D, E$ are new bases for $V, W$, respectively. Let $P$ be the transition matrix from $B$ to $D$, $Q$ that from $C$ to $E$. Then

$$A_{DE} = QA_{BC}P^{-1}.$$ 

Proof.
**Definition.** Let $X, Y$ be square matrices of the same size. If there exists an invertible matrix $P$ such that

$$Y = PXP^{-1}$$

then we call $X$ similar to $Y$.

Two matrices $X, Y$ are similar iff they are matrices for the same linear transformation wrt different bases.

**Theorem 7** If $A$ and $B$ are similar matrices, then they have the same characteristic polynomial.

**Theorem 8** Let $B, C, D$ be bases for $U, V, W$ and $L: V \to W$ and $K: U \to V$ linear transformations with matrices $M, N$, respectively. Then $L \circ K$ has matrix $MN$ wrt $B, D$. 
Proof.

\[
\begin{array}{ccc}
U & \xrightarrow{K} & V \\
\downarrow & & \downarrow \\
\mathbb{R}^m & \xrightarrow{N} & \mathbb{R}^n \xrightarrow{M} \mathbb{R}^p
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{[\cdot]_B} & \mathbb{R}^n & \xrightarrow{[\cdot]_C} & \mathbb{R}^p \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^m & \xrightarrow{[\cdot]_D} & \mathbb{R}^p
\end{array}
\]
5.3 The Dimension Theorem

Recall kernel, range. Examples from geometry, differentiation.

**Theorem 10 (Dimension Theorem)** Let $L : V \to W$ be a linear transformation. If $\dim(V)$ is finite then

$$\dim(\text{range}(L)) + \dim(\ker(L)) = \dim(V).$$

**Proof** - take coordinates!

**Corollary 11** If $L : V \to W$ and $V$ is finite-dimensional, then $\dim(\ker(L)) \leq \dim(V)$ and $\dim(\text{range}(L)) \leq \dim(V)$.

**Note** - $\dim(W)$ can be greater than $\dim(V)$. 
Find (basis of) \( \ker(L) \)

1. Find a matrix \( A \) for \( L \).

2. Perform row reduction, obtain \( R \).

3. For each independent variable \( x_i \) in \( Rx = 0 \), find a solution \( v_i \) with this variable = 1, other independents = 0.

4. The vectors \( v_i \) are coordinate vectors of a basis for \( \ker(L) \).
Find (basis of) range(\(L\))

1. Find a matrix \(A\) for \(L\).

2. Perform row reduction, obtain \(R\).

3. The pivot columns are coordinate vectors of a basis for range(\(L\)).

**Theorem 12** Let \(L : V \rightarrow W\) with \(V, W\) finite-dimensional. Let \(A\) be a matrix for \(L\) wrt some bases. Then

\[
\begin{align*}
(1) \quad \dim(\text{range}(L)) &= \text{rank}(A), \\
(2) \quad \dim(\ker(L)) &= \dim(V) - \text{rank}(A).
\end{align*}
\]

**Corollary 13** For any matrix \(A\),

\[\text{rank}(A) = \text{rank}(A^T).\]
5.4 Isomorphisms

**Definition.** A function (map, transformation) 
$L : V \to W$ is called **one-to-one** (injective) if for all $x_1 \neq x_2 \in V$

$$L(x_1) \neq L(x_2).$$

It is called **onto** (surjective) if $\text{range}(L) = W$.

$L$ is one-to-one iff (contrapositive!)

$$L(x_1) = L(x_2) \Rightarrow x_1 = x_2.$$

**Theorem 14** *Suppose $L : V \to W$ is a linear transformation. Then $L$ is one-to-one iff $\ker(L) = \{0\}$.***

**Definition.** A linear transformation $L : V \to W$ is called **invertible** iff there exists another transformation $K : W \to V$ such that $L \circ K = I_W$ and $K \circ L = I_V$. 
Theorem 15 If $L : V \rightarrow W$ is an invertible linear transformation, then the transformation $K$ above is unique, and it is also a linear transformation. It is denoted by $L^{-1}$.

Theorem 16 If $V, W$ are finite-dimensional then $L : V \rightarrow W$ is invertible iff the matrix for $L$ wrt any bases for $V, W$ is invertible.

Definition. A linear transformation $L : V \rightarrow W$ is called an isomorphism if it is both one-to-one and onto. Two vector spaces are called isomorphic if there exists an isomorphism between them.

Theorem 17 $L$ is an isomorphism iff $L$ is invertible.

Theorem 18 Suppose $V, W$ are finite-dimensional. $L$ is an isomorphism iff $\dim V = \dim W$ and either $\ker(L) = \{0\}$ or $\text{range}(L) = W$. 
Theorem 19 Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.
5.5 Diagonalization

$L$ is always a linear operator $V \to V$.

**Definition.** We call $v$ an **eigenvector of** $L$ corresponding to $\lambda$ if $v \neq 0$ and $L(v) = \lambda v$. We call $\lambda$ an **eigenvalue of** $L$ if there exists an eigenvector corresponding to $\lambda$.

For $V$ finite-dimensional, eigenvalues of $L$ are exactly the eigenvalues of any matrix of $L$.

Eigenvectors of $L$ have $B$-coordinates which are eigenvectors of $A_{BB}$.

**Definition.** For any eigenvalue $\lambda$ of $L$, define the **eigenspace of** $\lambda$, written $\text{Eig}_\lambda$, to be all vectors $v$ such that $L(v) = \lambda v$.

Example. $L(A) = A + A^T$. 
**Definition.** Let $A$ any matrix for $L$. Define the characteristic polynomial of $L$ to be that of $A$,

$$p_L(x) = p_A(x).$$

**Definition.** $L$ is called diagonalizable iff any matrix $A$ for $L$ is diagonalizable. Equivalently, some matrix for $L$ is diagonal.

**Theorem 20** $L$ is diagonalizable iff $V$ has a basis consisting of eigenvectors of $L$.

**Theorem 21** Suppose $\lambda_1, \ldots, \lambda_t$ are distinct eigenvalues of $L$, and $v_i$ is an eigenvector corresponding to $\lambda_i$ for $i = 1, \ldots, t$. Then $v_1, \ldots, v_t$ are linearly independent.

Proof - induction over $k = 1, \ldots, t$. 

Example. Consider $L$ given by

$$A = \begin{bmatrix}
31 & -14 & -92 \\
-50 & 28 & 158 \\
18 & -9 & -55
\end{bmatrix}.$$  

Here, $p_A(x) = (x + 1)(x - 2)(x - 3)$. Eigenvectors are eg $[2, -2, 1], [10, 1, 3], [1, 2, 0]$.

**Corollary 22** If $L$ has $\text{dim}(V)$ distinct linear eigenvalues, then $L$ is diagonalizable.

**Theorem 23** Suppose $\lambda_1, \ldots, \lambda_t$ are distinct eigenvalues of $L$ and $B_1, \ldots B_t$ are bases for the respective eigenspaces. Then the $B_i$ are pairwise disjoint, and their union is linearly independent.

Proof. Suppose the list of all vectors in $B_1, \ldots B_t$ is linearly dependent (this includes possible overlaps). Suppose $k$ is chosen such that $T = \{v_1, \ldots v_{k-1}\}$ is linearly dependent but $T \cup \{v_k\}$
is not. So there is a linear combination of $T$ giving $v_k$, 

$$\sum_{i=1}^{k-1} a_i v_i = v_k. \quad (7)$$

Apply $L$ to this equation which gives 

$$\sum_{i=1}^{k-1} a_i \lambda_i v_i = \lambda_k v_k. \quad (8)$$

Multiply Equation (7) by $\lambda_k$ and subtract from (8). 

$$\sum_{i=1}^{k-1} a_i (\lambda_i - \lambda_k) v_i = 0 \quad (9)$$

Now every coefficient here must be zero. So either $\lambda_i = \lambda_k$ or $a_i = 0$. When $\lambda_i = \lambda_k$, $v_k$ belongs to the same basis of $E_{\lambda_k}$ as all these $v_i$, so it is linearly independent from these too. Looking again at (7), these coefficients $a_i$ are zero as well.
Example. Consider

\[
A = \begin{bmatrix}
-4 & 7 & 1 & 4 \\
6 & -16 & -3 & -9 \\
12 & -27 & -4 & -15 \\
-18 & 43 & 7 & 24
\end{bmatrix}.
\]

Eigenvalues are $-1, 2, 0$. The eigenvalue 1 has an eigenspace of dimension 2.

**Definition.** The dimension $\dim(\text{Eig}_\lambda)$ is called the geometric multiplicity of $\lambda$, written $g_\lambda$. The exponent of $x - \lambda$ in the characteristic polynomial $p_L$ is called the algebraic multiplicity of $\lambda$, written $a_\lambda$.

**Theorem 24** For all eigenvalues $\lambda$ of $L$,

\[
1 \leq g_\lambda \leq a_\lambda.
\]