1. Linear Dependence and Linear Independence

**Definition.** The (indexed) set of vectors \( \{v_1, \ldots, v_k\} \) is **linearly independent** if whenever \( c_1, \ldots, c_k \) are coefficients such that \( c_1 v_1 + \cdots + c_k v_k = 0 \), then \( c_1 = \cdots = c_k = 0 \).

The set of vectors \( \{v_1, \ldots, v_k\} \) is **linearly dependent** if it is not linearly independent.

2. Some Exercises

(1) Express the definition of linear independence in symbolic form. Use quantifiers.
(2) Express the negation of “\( \{v_1, \ldots, v_k\} \) is linearly independent” in symbolic form. (The negation is, of course, another way to say “\( \{v_1, \ldots, v_k\} \) is linearly dependent.”) Rewrite your symbolic statement as an English sentence.
(3) Is the empty set linearly independent?
(4) Is the one-element set \( \{0\} \) linearly independent?
(5) If \( v \neq 0 \), is the one-element set \( \{v\} \) linearly independent?
(6) Prove: Any set \( \{v_1, \ldots, v_k\} \) containing \( 0 \) is linearly dependent.
(7) Suppose a two-element set \( \{u, v\} \) is linearly dependent. Must \( u \) and \( v \) be scalar multiples of each other?
(8) Let \( \mathcal{S} = \{u, v, w\} \) be a three-element set. Suppose no element of \( \mathcal{S} \) is a scalar multiple of any other. Must \( \mathcal{S} \) be linearly independent?