Problem 1  a) Find the general solution of
\[ x'' - 2x' + 10x = 0 \]
b) Find a particular solution of
\[ x'' - 2x' + 10x = 50 + e^t. \]  \hspace{1cm} (1)
c) What is the general solution of Equation (1)?  \hspace{1cm} (5+5+5=15 points)

Solution.  a) We look at the auxiliary equation
\[ r^2 - 2r + 10 = 0 \]
which has solutions \( r = 1 \pm 3i \), corresponding to the general solution
\[ x_c(t) = e^t(c_1 \cos 3t + c_2 \sin 3t). \]
b) There is no overlap of terms in \( x_c \) with terms 50 and \( e^t \) on the right-hand side, so we try
\[ x_p = A + Be^t. \]
We substitute \( x'_p \) and \( x''_p \) into the original equation, getting
\[ Be^t - 2Be^t + 10(A + Be^t) = 50 + e^t. \]
This means \( 10A = 50 \) and \( 9B = 1 \), so we put \( A = 5 \) and \( B = \frac{1}{9} \), and get the particular solution
\[ x_p(t) = 5 + \frac{e^t}{9}. \]
c) We just add up,
\[ x(t) = x_c(t) + x_p(t) = e^t(c_1 \cos 3t + c_2 \sin 3t) + 5 + \frac{e^t}{9}. \]
Problem 2 Set up a trial solution for (do NOT solve) (10 points)

\[ y'' + 8y' + 15y = 4te^{-5t} + \cos(3t). \]

Solution. We do need to check the auxiliary equation

\[ r^2 + 8r + 15 = (r + 3)(r + 5) = 0 \]

so \( r = -3 \) and \( r = -5 \) are roots, and the complementary function is

\[ y_c = c_1e^{-3t} + c_2e^{-5t}. \]

We notice the overlap with \( e^{-5t} \) on the right, so we need to have an extra factor of \( t \) for that, putting

\[ y_p(t) = t(A + Bt)e^{-5t} + C\cos 3t + D\sin 3t. \]

Problem 3 A cart of mass 9 kg is connected to the wall by a spring with stiffness \( k = 36 \) N/m. It moves horizontally without friction.

a) Let the displacement of the cart from rest position be \( x(t) \). Determine the equation of motion for \( x(t) \).

b) Given that \( x(0) = 5 \) and \( x'(0) = 0 \), find \( x(t) \) for arbitrary time \( t \). (5+5 = 10 points)

Solution. a) The equation is simply

\[ 9x'' + 36x = 0. \]

b) The auxiliary equation

\[ 9r^2 + 36 = 0 \]

has roots \( r = \pm 2i \), so a general solution is

\[ x(t) = c_1 \cos 2t + c_2 \sin 2t. \]

The initial conditions tell us \( c_1 = 5 \) and \( c_2 = 0 \).

Problem 4 Find the general solution of (10 points)

\[ t^2x'' - 5tx' + 8x = 0. \]
Solution. This is a Cauchy-Euler equation. We form the characteristic equation
\[ r(r - 1) - 5r + 8 = 0 \]
which has roots \( r = 2 \) and \( r = 4 \), leading to solutions \( t^2 \) and \( t^4 \). The general solution is then
\[ x(t) = c_1 t^2 + c_2 t^4. \]

Problem 5 A shell of mass 2 kg is shot upward with an initial velocity of 120 m/s, then allowed to fall under the influence of gravity. Assume that the force of air resistance is \(-8v\), where \( v \) is the velocity of the shell. If the shell is 100 m above the ground at time \( t = 0 \), determine when it will hit the ground, assuming that the exponential term has died off. Use the value \( g = 10 \) N/kg for the magnitude of the acceleration due to gravity. (15 points)

Solution. You can think of Newton’s Second Law, \( ma = F \) with the total force being gravity + friction. But in standard form, you will want to rewrite this as
\[ 2v' + 8v = -2g, \quad \text{or} \quad v' + 4v = -10. \]
So the integrating factor is \( e^{4t} \), and
\[ e^{4t}v = -10 \int e^{4t} dt = -\frac{5}{2} e^{12t} + C, \]
which we solve for \( v \),
\[ v = -\frac{5}{2} + Ce^{-4t}. \]
From \( v(0) = 120 \), we get
\[ C = 120 + \frac{5}{2} = \frac{245}{2}. \]
Then the position is (note that we have to use a new variable instead of \( t \), because \( t \) is already in use as the upper limit of the integral)
\[ x(t) = \int_0^t -\frac{5}{2} + \frac{245 e^{-4s}}{2} \, ds = -\frac{5}{2} t - \frac{245(e^{-4t} - 1)}{8} + D. \]
From \( x(0) = 100 \), we determine \( D = 100 \). The exponential becomes negligible very soon, and if we neglect it, we can solve
\[ 0 = x(t) \approx -\frac{5}{2} t + \frac{245}{8} + 100 \]
which gives the time when the shell hits the ground as

\[ t = \frac{209}{4} = 52.25 \]  

(in seconds).