Math 267 - Exam 1A - solutions

Problem 1 Have a look at the whole exam paper. Which of the numbered differential equations is/are linear? (just list their numbers).
Solution. Only (4), (5).

Problem 2 For which of the following initial value problems does Theorem 1.2.1 (section 1.2 of Chapter 1) guarantee existence and uniqueness of a solution, at least in a small rectangle? Justify your answers briefly.

\[
\begin{align*}
y' &= \frac{x^2 - 4}{y - 1} & y(2) &= 1 \\
y' &= \frac{x^2 - 4}{y - 1} & y(2) &= 0 \\
y' &= 8 - \sqrt{y} & y(5) &= 0 \\
y' &= \frac{x^2 - y}{x - 1} & y(2) &= 4
\end{align*}
\]

Solution. Theorem 1 applies to (2) and (4) only. Checks for (1): \( f(x, y) = \frac{x^2 - 4}{y - 1} \) is not defined for \( y = 1 \). So it can’t even have a partial derivative there.
Checks for (2): \( f(x, y) = \frac{x^2 - 4}{y - 1} \) is defined and differentiable near \((2, 0)\), so continuous. The same is true for \( \partial f/\partial y \).
Checks for (3): \( f(x, y) = 8 - \sqrt{y} \) is continuous everywhere. But the derivative \( \partial f/\partial y = -y^{-1/2}/3 \) is not defined for \( y = 0 \), so not continuous. Theorem 1.2.1 does not apply.
Checks for (4): \( f(x, y) = \frac{x^2 - y}{x - 1} \) is continuous near \( x = 2, y = 4 \). The same is true for the partial derivative \( \partial f/\partial y = -1/(x - 1) \).

Problem 3 Consider the differential equation

\[ xy' + 3y = 4x + \frac{\cos x}{x^2} \]

a) Find the general solution. Write down intermediate steps explaining the method you are using - the result alone is not enough.
b) Solve the initial value problem consisting of Equation (5) and \( y(\pi) = 5 \).
c) What can you say about existence and uniqueness of the solution of the initial value problem (IVP) involving (5) and the condition \( y(\pi) = 5 \)? What is the biggest interval where you can be sure that solutions of such an IVP are defined?

**Solution.**  
a) This equation is linear. But to apply the recipe, you have to divide by \( x \) first (\( y' \) has to be isolated)! You get

\[
y' + \frac{3y}{x} = 4 + \frac{\cos x}{x^3}.
\]

Then the recipe tells us to put

\[
\mu(x) := e^{\int \frac{3}{x} \, dx} = |x|^3,
\]

to multiply equation (6) by \( \mu(x) \) and then integrate. We write \( \text{sgn}(x) = -1 \) or 1, depending on whether \( x < 0 \) or \( x > 0 \), and integrate

\[
|x|^3 y = \text{sgn}(x) \int 4x^3 + \cos x \, dx = \text{sgn}(x)(x^4 + \sin(x)) + C.
\]

Solve for \( y \) – the sign \( \text{sgn}(x) \) cancels out, and we get

\[
y(x) = x + \frac{\sin(x) + C}{x^3}.
\]

Alternatively, we could say that for this initial value problem, always \( x > 0 \) since we are supposed to look near \( x = \pi \).

b) Substitute \( x = \pi \), \( y = 5 \) into equation (8) to get \( 5 = \pi + C/\pi^3 \), so \( C = 5\pi^3 - \pi^4 \) and

\[
y(x) = x + \frac{\sin(x) + 5\pi^3 - \pi^4}{x^3}
\]

c) The existence / uniqueness of this solution is guaranteed for all \( x > 0 \) and all \( x < 0 \) because of the existence/uniqueness theorem for linear equations.

**Problem 4** Consider the differential equation

\[
(2xy + 5)dx + (x^2 - 3y^2)dy = 0.
\]

a) Show that this equation is exact by performing the test for exactness.

b) Find an implicit solution of equation (9).

**Solution.**  
a) Define \( M(x,y) := 2xy + 5 \) and \( N(x,y) := x^2 - 3y^2 \). Then

\[
M_y = 2x = N_x
\]
so the equation is exact.

b) Put $F(x, y) := \int M \, dx = x^2y + 5x + C(y)$. Compare $F_y$ to $N$:

\[
F_y = x^2 + C'(y) \\
N = x^2 - 3y^2
\]

So $C'(y) = -3y^2$, and $g(y) = -y^3 + D$. The implicit solution is $F(x, y) = C$ where

\[
F(x, y) = x^2y + 5x - y^3 + D.
\]

**Problem 5** Consider the differential equations

\[
a) \ y' = \frac{x + y + 2}{y + 2} \quad (11) \\
b) \ y' + 2xy = 6xy^5 \quad (12) \\
c) \ x^3y' = 2xy^2 + y^3 \quad (13)
\]

Give substitutions $v = v(x, y)$ for each of them transforming them into separable or linear equations. Write down the definition of $v$ and the resulting differential equation for $v$ which is separable/linear - do NOT solve them.

**Solution.**

a) Here, we need to apply a shift by a constant, $u = y + 2$ to make the equation homogeneous,

\[
u' = u' = \frac{x + u}{u}
\]

and then the recipe for homogeneous equations, suggesting $v = u/x = (y + 2)/x$ so

\[
u' = \frac{1 + v}{v} \\
v' = \frac{u' - v}{x} = \frac{1 + v}{x} - \frac{v}{x}
\]

which is separable. Stop right there!

b) This equation is already separable! you could say $v = y$, and

\[
y' = 2x(3y^5 - y), \quad \frac{1}{3y^5 - y} \, dy = \frac{1}{2x} \, dx.
\]
Alternatively, you can use that it is a Bernoulli equation with \( n = 5 \), so use \( v = y^{-4} \). You get

\[
\frac{dv}{dx} = -4y^{-5}(6xy^5 - 2xy) = -24x + 8xv
\]

which is linear. Stop!

c) This is a homogeneous equation, so use \( v = y/x \). Then

\[
y' = 2v^2 + v^3
\]

and consequently,

\[
\frac{dv}{dx} = \frac{y' - v}{x} = \frac{2v^2 + v^3 - v}{x}
\]

which is separable - stop.