C Roettger, Fall 16

Math 265 – General Substitutions (Coordinate Transformations) for Integrals

We study coordinate transformations of the form

$$(x, y) = T(u, v)$$

(2 variables input, 2 variables output). For integrals, this can potentially make them easier to solve.

Consider the following integration problem.
Find the average of the function

$$f(x, y) = (4x - 5y)e^{3x+2y}$$

over the parallelogram with vertices $A = (0, 0)$, $B = (10, 8)$, $C = (16, -1)$, $D = (6, -9)$.
Solution We want to use a transformation $T(u, v)$ such that $u = 3x + 2y$ and $v = 4x - 5y$ which will definitely make the integrand simpler. But we have to find that transformation first! So solve the above two equations for x and y by doing

$$\begin{aligned} 5u + 2v &= 23x \\ \frac{5u + 2v}{23} &= x \\ 4u - 3v &= 23y \\ \frac{4u - 3v}{23} &= y. \end{aligned}$$

Now we compute the Jacobian,

$$J(u, v) = \begin{vmatrix} \frac{5}{23} & \frac{2}{23} \\ \frac{4}{23} & -\frac{3}{23} \end{vmatrix} = \frac{1}{23},$$

and the domain E such that $T(E)$ is the given parallelogram. Since T is a linear transformation, E will also be a parallelogram (straight lines get transformed into straight lines by T). We just need to apply $u = 3x + 2y$ and $v = 4x - 5y$ to each of the four points A, B, C, D knowing that they will be the vertices of E. For A, we get $\alpha = (0, 0)$. For B, we get

$$\beta = (3 \cdot 10 + 2 \cdot 8, 4 \cdot 10 - 5 \cdot 8) = (46, 0)$$

meaning $T(\alpha) = A$ and $T(\beta) = B$. Similarly, we compute

$$\gamma = (46, 69), \qquad \delta = (0, 69).$$

Note that as vectors, $C = B + D$ and $\gamma = \beta + \delta$ (the transformation T preserves these relationships). It turns out that the parallelogram E is actually a rectangle, with sides parallel to coordinate axes! this is an unexpected windfall (OK, your calculus instructor tweaked the problem to make it happen, but you could not tell from the outset).
Now all the ingredients are in place.

$$\begin{aligned} \iint_D f(x,y)\,\mathrm{d}A(x,y) &= \iint_E ve^u J(u,v)\,\mathrm{d}A(u,v) \\ &= \frac{1}{23}\int_0^{46}\int_0^{69} ve^u\,\mathrm{d}v\,\mathrm{d}u = \frac{69^2(e^{46}-1)}{46}. \end{aligned}$$

Similarly, for the area we get

$$\begin{aligned} \iint_D 1\,\mathrm{d}A(x,y) &= \iint_E J(u,v)\,\mathrm{d}A(u,v) \\ &= \frac{1}{23}\int_0^{46}\int_0^{69} 1\,\mathrm{d}v\,\mathrm{d}u = \frac{46\cdot 69}{23} = 138. \end{aligned}$$

Then the average value of $f(x, y)$ on the original parallelogram is

$$\bar{f} = \frac{69^2(e^46-1)}{46\cdot 138} = \frac{3(e^{46}-1)}{4}.$$

Think for a moment how hard it would have been to use x, y to evaluate this integral!