Problem 1  Consider the integral
\[ \int_{3}^{\infty} \frac{1}{x^3 - 4x} \, dx. \]

a) Show that this integral converges by comparison to the integral over \(1/x^3\).

b) Compute the exact value of this integral (hint: partial fractions).

Solution  a) The limit of the ratio of the given function and \(1/x^3\) is One,
\[ \lim_{x \to \infty} \frac{x^3}{x^3 - 4x} = 1. \]

Also, the integral over \(1/x^3\) converges,
\[ \int_{3}^{\infty} \frac{1}{x^3} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{2x^2} \right]_{3}^{b} = \frac{1}{2}. \]

Therefore the given integral converges by the LCT.

If you want to use the DCT you have to deal with the problem that \(1/x^3\) is smaller, not bigger, than the given function. You could eg argue that
\[ \frac{1}{x^3 - 4x} \leq \frac{2}{x^3} \]
at least for \(x\) large enough. But the DCT is not so convenient to use here.

b) Computing the exact value requires more effort. We factorize the denominator
\[ x^3 - 4x = x(x - 2)(x + 2) \]
and set up
\[ \frac{1}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 2}, \]
then multiply both sides by the denominator and simplify
\[ 1 = A(x^2 - 4) + Bx(x + 2) + Cx(x - 2). \]
We set \( x = 0 \) here, which gives \( A = -1/4 \). Then \( x = 2 \) gives \( B = 1/8 \) and \( x = -2 \) gives \( C = 1/8 \). Comparing like powers (the standard approach for quadratic factors or repeated linear factors) could also be done, but takes a little longer. Whichever way you do it, you need to get to
\[
\frac{1}{x^3 - 4x} = -\frac{1}{4x} + \frac{1}{8(x - 2)} + \frac{1}{8(x + 2)}.
\]
Now we can integrate.

\[
\int_3^b \frac{1}{x^3 - 4x} \, dx = \frac{1}{8} \int_3^b -\frac{2}{x} + \frac{1}{x - 2} + \frac{1}{x + 2} \, dx
\]
\[
= \frac{1}{8} \left[ -2 \ln x + \ln |x - 2| + \ln |x + 2| \right]_3^b
\]
\[
= \frac{1}{8} \left[ -2 \ln b + \ln |b - 2| + \ln |b + 2| + 2 \ln 3 - \ln 5 \right]
\]
\[
= \frac{1}{8} \ln \left( \frac{b^2 - 4}{b^2} \right) + \frac{1}{8} \ln \left( \frac{9}{5} \right)
\]
We need to take the limit of this for \( b \to \infty \). But since
\[
\lim_{b \to \infty} \frac{b^2 - 4}{b^2} = 1,
\]
the logarithm of this has limit zero. So \( \frac{1}{8} \ln \left( \frac{9}{5} \right) \) is the value of the integral.

**Problem 2** Find the limits of the following sequences or explain why they do not exist.

\[
a_n = \frac{n^4 - \sin n}{6n^4 + \cos(n)}
\]
\[
b_n = 2^{-n} + (-1)^n
\]
\[
c_n = (4 + \sqrt{n})^{2/n}
\]

**Solution** a) Divide numerator and denominator by highest power of \( n \) present \((n^4)\), the limit is \( 1/6 \) (can also mention the Sandwich Theorem on \( \sin n/n^4 \) and \( \cos n/n^4 \)).

b) This limit does not exist. The limit of \( 2^{-n} \) is zero, \((-1)^n\) oscillates between
c) One way to do this is to begin with
\[ \ln c_n = \frac{2}{n} \ln(4 + \sqrt{n}). \]

Then use l'Hopital's Rule on
\[ \frac{f(x)}{g(x)} = \frac{2 \ln(4 + \sqrt{x})}{x} \]
so the limit of this for \( x \to \infty \) is
\[ \lim_{x \to \infty} \frac{2}{2\sqrt{x}(4 + \sqrt{x})} = 0. \]

Therefore \( \lim_{n \to \infty} c_n = e^0 = 1. \)

As an alternative, we could have used the Sandwich Theorem with
\[ 1 \leq \sqrt{c_n} \leq n^{1/n} \]
which shows \( \lim_{n \to \infty} \sqrt{c_n} = 1. \)

**Problem 3** Show that the integral
\[ \int_2^\infty \frac{1}{x \ln x} \, dx \]
diverges. Then explain what this means for the series
\[ \sum_{n=2}^\infty \frac{1}{n \ln n} \]

**Solution** By definition, this integral equals the limit for \( b \to \infty \) of
\[ \int_2^b \frac{1}{x \ln x} \, dx = \int_{\ln 2}^\infty \frac{1}{u} \, du. \]
(with a substitution \( u = \ln x, \ du = \frac{1}{x} \, dx \)). The last integral evaluates to
\[ [\ln u]_{\ln 2}^{\ln b} = \ln b - \ln 2 \]
and this tends to $\infty$ as $b \to \infty$. For the given series, this just means that it also diverges.

**Problem 4** For each of the following series, compute the sum or show that it diverges using an appropriate test. Name the test used and give details, e.g. the integral you considered for the Integral Comparison Test.

a) $\sum_{k=0}^{\infty} \frac{5^k}{3^{2k}}$

b) $\sum_{m=1}^{\infty} \left( \frac{3}{m! + 4} - \frac{3}{(m+1)! + 4} \right)$

c) $\sum_{n=1}^{\infty} \frac{2n + 1}{n^2 + n}$

**Solution**

a) Is a geometric series with $a = 1$ and $r = 5/9$. So the sum of the series is

$$\frac{1}{1 - 5/9} = \frac{9}{4}.$$

b) This series is telescoping. The terms obviously converge to zero, so the limit is

$$\frac{3}{1! + 4} = \frac{3}{5}.$$  

c) This series diverges. You can use the LCT with $b_n = 1/n$, or the integral comparison test with $f(x) = \frac{2x + 1}{x^2 + x}$ integrated from 1 to $\infty$. That integral diverges, which you can see either by substitution $u = x^2 + x$, $du = (2x + 1)dx$, or by a Limit comparison test with $g(x) = 1/x$.

**Problem 5** In a half-circle of radius 1, we inscribe two smaller half-circles of radius 1/2. Continue the process with each of the smaller half-circles indefinitely. Every new generation of half-circles with radius $2^{-k}$ gets a new layer of paint, with constant density $\delta = 1.2^k$. The resulting figure contains infinitely many half-circles. Below is a plot where only the first four generations of half-circles are shown.

a) Find the total amount of paint needed for the resulting figure (for each
generation, you need $\delta$ times the sum of areas of all half-circles in that generation).
b) What is the total length of the arcs of circles in the resulting figure?

**Solution**

a) There are $2^k$ half-circles in generation $k$, with radius $2^{-k}$. The area of each is $\pi 2^{-2k-1}$. Summed over $k$, the total amount of paint is

$$m = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1.2^k}{2^k} = \frac{\pi}{2(1 - 1.2/2)} = \frac{5\pi}{4}.$$  

The amount of paint needed is then $\frac{5\pi}{4}$.

b) The half-arc for a circle in generation $k$ has length $2^{-k}\pi$. There are $2^k$ circles in that generation, so each generation has a total arc length of $\pi$. Summed over all generations, this diverges to infinity.