Math 151 - Practice Exam 3 - solutions

**Problem 1** Find the absolute maximum and absolute minimum values of the function

\[ f(x) = x^4 - 4x^3 + 6x^2 \]

on the interval \([-2, 2]\). Indicate at which \(x\)-values each of these extrema occur.

**Solution.** We need to calculate

\[ f'(x) = 4x^3 - 12x^2 + 12x \]

and solve \(f'(x) = 0\). Factorize \(f'(x)\) as

\[ f'(x) = 4x(x^2 - 3x + 3) \]

is zero at \(x = 0\) only (use the quadratic formula or complete the square to decide that \(x^2 - 3x + 3 = 0\) has no solution). So we have only three critical points \(-2, 0, 2\), and the table of values of \(f(x)\) at critical points is

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-2)</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>72</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

and therefore the maximum of \(f(x)\) on \([-2, 2]\) is 72, occurring at \(x = -2\). The minimum is 0, occurring at \(x = 0\).

**Problem 2** Consider the function

\[ f(x) = x^3 - 6x^2 + 1. \]

(a) Determine the sign pattern of \(f'(x)\).
(b) Find the intervals on which \(f(x)\) is increasing.
(c) Find the intervals on which \(f(x)\) is decreasing.
(d) Find all \(x\)-values where \(f(x)\) has a local maximum.
(e) Find all \(x\) values where \(f(x)\) has a local minimum.
(f) Find the intervals on which \(f(x)\) is concave up.
(g) Find the intervals on which \( f(x) \) is concave down.

**Solution.**

(a) We need to calculate

\[
f'(x) = 3x^2 - 12x
\]

and solve \( f'(x) = 0 \). The solutions are \( x = 0, 4 \). From \( f'(-1) > 0 \), \( f'(1) < 0 \), and \( f'(5) > 0 \) we see that the sign pattern of \( f'(x) \) is

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of ( f' )</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>( f ) is</td>
<td>incr.</td>
<td>decre.</td>
<td>incr.</td>
</tr>
</tbody>
</table>

(b) From the table in (a), \( f \) is increasing in \((-\infty, 0]\) and in \([4, \infty)\).
(c) From the table in (a), \( f \) is decreasing in \([0, 4]\). If you put open intervals in part (b) or (c) then we’ll not take marks off.
(d) From the table in (a), \( f \) has a local maximum at \( x = 0 \).
(e) From the table in (a), \( f \) has a local minimum at \( x = 4 \).
(f) First, \( f''(x) = 6x - 12 \). So \( f''(x) < 0 \) for \( x < 2 \) and \( f''(x) > 0 \) for \( x > 2 \). So \( f(x) \) is concave down for \( x < 2 \).
(g) \( f(x) \) is concave up for \( x > 2 \) - work done in (f) is enough for both parts! (if it was a more complicated function, we could fall back on the sampling method to determine the sign of \( f''(x) \)).

**Problem 3** Suppose the demand \( x \) (= number of units sold) of Halloween Slime Cookies\textsuperscript{TM} is linked to unit price \( p \) by

\[
p = \sqrt{500 - x^2}
\]
(a) Find the revenue $R(p)$ as a function of $p$.
(b) Find the price $p$ which gives you maximal revenue.

**Solution.** (a) First, we solve the demand equation for $x$ to get

$$x = \sqrt{500 - p^2}$$

and then use $R = xp$ to get

$$R(p) = p\sqrt{500 - p^2}.$$

(b) Obviously, only $0 \leq p \leq \sqrt{500}$ are legitimate values for $p$. There, we differentiate

$$R'(p) = \sqrt{500 - p^2} + p \cdot \frac{-2p}{2\sqrt{500 - p^2}} = \sqrt{500 - p^2} - \frac{p^2}{\sqrt{500 - p^2}}.$$

Solving $R' = 0$ gives

$$500 - 2p^2 = 0$$

with solutions $p = \pm \sqrt{250} = \pm 5\sqrt{10}$. Discarding the negative answer, the critical points are $p = 0, \sqrt{250}, \sqrt{500}$ (including endpoints). Note that $p = \sqrt{500}$ is also a point where $R(p)$ is not differentiable (has a vertical tangent line), but we had it on the list as endpoint anyway. Clearly, $R(0) = R(\sqrt{500}) = 0$ is not the maximum, so $p = \sqrt{250}$ will give maximal revenue (we could have made a table like in the previous problem, too).

**Problem 4** If each edge of a cube is increasing at a rate of 3 centimeters per second, how fast is the volume increasing when $x$, the length of an edge, is 15 centimeters long?

**Solution.** Let $V$ be the volume of the cube, so $V = x^3$ at all times $t$. Therefore

$$V'(t) = 3x(t)^2 x'(t)$$

and substituting $x = 15$ and $x' = 3$, we get $V'(t) = 2025$ (in cubic centimeters per second).

**Problem 5** Suppose gross domestic product (GDP) $G$ and population $P$ of a country are related by the equation

$$G^2 - 0.3GP + P^{2/3} = 17.$$
Both $G$ and $P$ are functions of time $t$. At a time when $G = 5$ and $P = 8$ (in billions $\$ \text{ and millions of people, respectively}$), the GDP grows at a rate of $G'(t) = 0.06$ millions per year. Use related rates to find the corresponding rate of change for $P$.

**Solution.** We can differentiate both sides, like in the previous problem, using the Chain Rule.

$$2GG' - 0.3(G'P + GP') + \frac{2P'}{3P^{1/3}} = 0$$

Then we substitute $G = 5$, $P = 8$, and $G' = 0.06$ to get

$$0.6 - 0.3(0.48 + 5P') + \frac{2P'}{6} = 0.$$ 

Solving this for $P'$ (multiply by 3, collect all terms with $P'$ on the right side, divide by 3.5) gives

$$P' = \frac{1.368}{3.5} \approx 0.391$$

(in billions per year).

**Problem 6** Two quantities $x, y$ are related by

$$y^3 + xy = x^3 - x^2 - 1.$$ 

Suppose that is a function of $x$, ie $y = f(x)$ and that for $x = 2$, $y = 1$.

(a) Find $\frac{dy}{dx}$ at $x = 2$ using implicit differentiation.

(b) Set up an equation for the tangent line of the graph of $f(x)$ at $x = 2$.

(c) If $x$ changes to 2.03, find the approximation of $f(2.03)$ using your equation for the tangent line in (b). Note: more accurate approximations using your calculator’s SOLVE function will not receive credit.

**Solution.** (a) Differentiate both sides:

$$3y^2y' + y + xy' = 3x^2 - 2x$$

then collect all terms with $y'$ on the left side, factor out $y'$ and solve for $y'$:

$$y'(x) = \frac{3x^2 - 2x - y}{3y^2 + x}$$
(b) Use part (a) and plug in \( x = 2, \ y = 1 \) into the formula for \( y' \) to get
\[
y'(2) = \frac{12 - 4 - 1}{3 + 2} = \frac{7}{5}.
\]
Then the desired tangent line has slope \( \frac{7}{5} \), passes through \( (2, 1) \). It therefore has a point-slope equation
\[
y = \frac{7}{5}(x - 2) + 1.
\]
(c) Simply substitute \( x = 2.03 \) into the tangent line equation from part (b) to get
\[
y = \frac{7}{5}(2.03 - 2) + 1 = 1.042.
\]

**Problem 7** Find the indefinite integrals (= family of all antiderivatives) of
(a) \( f(x) = 15x^4 - 8x + 1 \).
(b) \( g(u) = 2u^{3/4} - 3 \).

**Solution.** Make sure to keep the same variable as in the problem, and to add on the integration constant at the end.

(a) \[ \int f(x) \, dx = 3x^5 - 4x^2 + x + C \]
(b) \[ \int g(u) \, du = \frac{8}{7}u^{7/4} - 3u + C. \]