Math 151 - Practice Exam 2 - solutions

Problem 1  Find $\frac{dy}{dx}$ where $y = x^2 + 10x + x^{7/4}$.

Solution. None of the big Rules needed. $f'(x) = 2x + 10 + \frac{7}{4}x^{3/4}$.

Problem 2  Consider the function

$$f(x) = \frac{x - 2}{3x - 1}.$$

a) Find $f'(2)$.

b) What is the slope-intercept equation of the tangent line to the graph of $f(x)$ at $x = 2$?

Solution.  a) We need the Quotient Rule with $u = x - 2$ and $v = 3x - 1$.

$$\frac{d}{dx} \frac{x - 2}{3x - 1} = \frac{(3x - 1) - 3(x - 2)}{(3x - 1)^2} = \frac{5}{(3x - 1)^2}.$$  

Then we substitute $x = 2$ to get $f'(2) = 5/25 = 1/5$.

b) Since $f(2) = 0$, we have

$$y = \frac{1}{5}(x - 2) + 0$$

which expands to the slope-intercept form $y = \frac{x}{5} - \frac{2}{5}$.

Problem 3  Consider the function

$$f(x) = x^5(2x^3 + 7\sqrt{x}).$$
a) Find \( f'(x) \) using the Rule for Products.
b) Expand the formula for \( f(x) \) and compute \( f'(x) \) using the Rule for powers of \( x \).

**Solution.**  

a) With \( u = x^5 \) and \( v = 2x^3 + 7\sqrt{x} \), we first compute \( u' = 5x^4 \) and \( v' = 6x^2 + 7/(2\sqrt{x}) \). Then

\[
f'(x) = x^5(6x^2 + 7/(2\sqrt{x})) + 5x^4(2x^3 + 7\sqrt{x}).
\]

b) We expand \( f(x) \) to

\[
f(x) = 2x^8 + 7x^{5.5}.
\]

Then we can apply the basic rules of 4.1 to get the same answer in a different form,

\[
f'(x) = 16x^7 + 38.5x^{4.5}.
\]

Yes, they agree.

**Problem 4**  

Suppose \( f(a) = 2, \ g(a) = 3, \ f'(a) = 5 \) and \( g'(a) = -1 \). Find the derivative of \( f(x)(g(x) - 4) \) at \( x = a \).

**Solution.**  

Use the product rule,

\[
\frac{d}{dx}[f(x)(g(x) - 4)] = f(x)g'(x) + f'(x)(g(x) - 4).
\]

Now we substitute \( x = a \) and the given information to get the value

\[
2(-1) + 5(3 - 4) = -7.
\]

**Problem 5**  

Let \( f(x) = x^3 - 9x^2 + 24x - 1 \).

Find all the points at which the graph of \( f \) has a horizontal tangent line.

**Solution.**

\[
f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x - 2)(x - 4).
\]

This is zero for \( x = 2 \) and \( x = 4 \), and these points are exactly where the graph of \( f \) has a horizontal tangent line.

**Problem 6**  

Find \( \frac{dy}{dx} \) where \( y = 2x^2 - 4x + 1 \) at \( x = a \) using a limit computation, NOT the Rules from Chapter 4. Then double-check using the
Rules from Chapter 4.

Solution. First,

\[ y' = \lim_{h \to 0} \frac{2(a + h)^2 - 4(a + h) + 1 - (2a^2 - 4a + 1)}{h} \]
\[ = \lim_{h \to 0} \frac{2(a^2 + 2ah + h^2 - a^2) - 4h}{h} \]
\[ = \lim_{h \to 0} \frac{4a + 2h - 4}{h} = 4a - 4. \]

And of course, this agrees with the Rules from 4.1.

Problem 7 Find \( D_t(m(t)) \) where \( m(t) = 0.02\sqrt{t} + 200 \) using a limit computation, NOT the Rules from Chapter 4. Then double-check using the Rules from Chapter 4. What is the instantaneous rate of change of \( m \) at \( t = 25 \)?

Solution.

\[ D_t m(t) = \lim_{h \to 0} \frac{0.02\sqrt{t + h} + 200 - (0.02\sqrt{t} + 200)}{h} \]
\[ = \lim_{h \to 0} \frac{0.02(\sqrt{t + h} - \sqrt{t})(\sqrt{t + h} + \sqrt{t})}{h(\sqrt{t + h} + \sqrt{t})} \]
\[ = \lim_{h \to 0} \frac{0.02(t + h - t)}{h(\sqrt{t + h} + \sqrt{t})} \]
\[ = \lim_{h \to 0} \frac{0.02}{\sqrt{t + h} + \sqrt{t}} = \frac{0.01}{\sqrt{t}}. \]

Then we can just substitute \( t = 25 \) to get \( D_t m(25) = 0.01/5 = 0.002 \).

Problem 8 Assume that the unit price of manufacturing \( x \) batches of paper towels is

\[ p(x) = 412 - \frac{1}{3}x^2 \]

(a) What is the revenue \( R(x) \) from selling these \( x \) batches?
(b) Find the marginal revenue \( R'(x) \) at \( x = 20 \).
(c) Use your answer in part (b) to approximate the additional revenue from selling 2 extra batches in addition to the 20 batches from part (b). The exact answer will not give credit.
(d) Find an equation for the tangent line to the graph of \( R(x) \) at \( x = 20 \). No need to simplify!
Solution. (a) \( R = xp = 412x - x^3/3. \)
(b) \( R'(x) = 412 - x^2, \) so \( R'(20) = 12. \)
(c) \( R(22) \approx R(20) + 2R'(20) = R(20) + 24, \) so 24 dollars additional revenue.
(d) \( y = 12(x - 20) + 8240 - 8000/3. \)

Problem 9  Assume that the unit price of manufacturing \( x \) batches of flibbertigibbets is
\[ p(x) = 20.1 - 0.02x \]
and the total cost is
\[ C(x) = 150 + 0.1x. \]

(a) What is the total profit \( P(x) \) from selling these \( x \) batches, and the average profit \( \bar{P}(x) \)?
(b) Find the marginal average profit at \( x = 20. \)
(c) For what value of \( x \) is the average profit \( \bar{P}(x) \) maximal?
(d) Find an equation for the tangent line to the graph of \( P(x) \) at \( x = 20. \) No need to simplify!

Solution. (a) \( P(x) = xp(x) - C(x) = 20x - 0.02x^2 - 150. \) So
\[ \bar{P}(x) = \frac{20x - 0.02x^2 - 150}{x} = 20 - 0.02x - \frac{150}{x} \]
(b) \( \bar{P}'(x) = -0.02 + 150/x^2, \) so \( \bar{P}'(20) = 0.355. \)
(c) We need to solve \( \bar{P}'(x) = 0, \) so \( 0.02x^2 = 150. \) After discarding the negative answer, this gives \( x = 50\sqrt{3}. \) OK, without a calculator we’d just stop here. On the real exam I will make sure the answers are nice integers.
(d) Read these questions carefully! this is about the ordinary profit again, not average profit. The derivative of that is
\[ P'(x) = 20 - 0.04x \]
so at \( x = 20, \) \( P'(20) = 19.2. \) Thus, the tangent line has equation \( y = 19.2(x - 20) + P(20) = 19.2(x - 20) + 242. \)