Math 151 - Exam 2A - solutions

Problem 1 Find $f'(t)$ for

$$f(t) = \frac{3t^2 + \sqrt{t}}{t^3}.$$

Solution. You can use the Quotient Rule. But it’s actually easier to first rewrite $f(t)$ as

$$f(t) = 3t^{-1} + t^{-2.5}$$

and then just use the Rules for powers:

$$f(t) = -3t^{-2} - 2.5t^{-3.5}.$$

Problem 2 Consider the function

$$f(x) = \frac{4x^2 - 1}{x + 3}.$$

a) Find $f'(-2)$.

b) What is the slope-intercept equation of the tangent line to the graph of $f(x)$ at $x = -2$?

c) For what value(s) of $x$ does the derivative $f'(x)$ not exist?

Solution.
a) We need the Quotient Rule with $u = 4x^2 - 1$ and $v = x + 3$.

\[
\frac{d}{dx} \frac{4x^2 - 1}{x + 3} = \frac{8x(x + 3) - (4x^2 - 1)}{(x + 3)^2} = \frac{4x^2 + 24x + 1}{(x + 3)^2}.
\]

The simplification is helpful for the rest of this problem. We substitute $x = -2$ to get $f'(-2) = -31$.

b) Since $f(-2) = 15$, we have

\[
y = -31(x + 2) + 15
\]

which expands to the slope-intercept form $y = -31x - 47$.

c) for $x = -3$ because the denominator is 0 there.

**Problem 3** Suppose $f(a) = 4$ and $f'(a) = 1/2$. Find the derivative of $x^3f(x)$ at $x = a$.

**Solution.** Use the product rule, with $u = x^3$ and $v = f(x)$ so $u' = 3x^2$ and $v' = f'(x)$, so

\[
\frac{d}{dx} x^3f(x) = 3x^2 f(x) + x^3 f'(x).
\]

Then substitute $x = a$ to get the derivative at $x = a$ equals $12a^2 + a^3/2$.

**Problem 4** Consider the function

\[
f(x) = (x^4 + 2x)(3x^2 + 1).
\]

a) Find $f'(x)$ using the Rule for Products.

b) Expand the formula for $f(x)$ and compute $f'(x)$ using the Rule for powers of $x$.

**Solution.** a) With $u = x^4 + 2x$ and $v = 3x^2 + 1$, you get $u' = 4x^3 + 2$ and $v' = 6x$, so

\[
f'(x) = (x^4 + 2x)(6x) + (4x^3 + 2)(3x^2 + 1).
\]

b) First, $f(x)$ is rewritten in expanded form as

\[
f(x) = 3x^6 + x^4 + 6x^3 + 2x
\]

and then we differentiate, using the rule for powers of $x$,

\[
f'(x) = 18x^5 + 4x^3 + 18x^2 + 2.
\]
If you expand the result from a) you’ll see that this agrees with part b).

**Problem 5** Let \( f(x) = x^3 - 9x^2 + 24x - 1 \).
Find all \( x \)-values at which the graph of \( f \) has a horizontal tangent line.

**Solution.** These are the values where \( f'(x) = 0 \). From

\[
f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x - 2)(x - 4),
\]
you see that this is zero for \( x = 2 \) and \( x = 4 \).

**Problem 6** Find \( \frac{dy}{dx} \) where \( y = 1 + 5\sqrt{x} \) at \( x = a \), using a limit computation, NOT the Rules from Chapter 4. Then double-check using the Rules from Chapter 4.

**Solution.** First,

\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{1 + 5\sqrt{a + h} - (1 + 5\sqrt{a})}{h}
\]

\[
= \lim_{h \to 0} \frac{5(\sqrt{a + h} - \sqrt{a})}{h}
\]

\[
= \lim_{h \to 0} \frac{5(\sqrt{a + h} - \sqrt{a})(\sqrt{a + h} + \sqrt{a})}{h(\sqrt{a + h} + \sqrt{a})}
\]

\[
= \lim_{h \to 0} \frac{5(a + h - a)}{h(\sqrt{a + h} + \sqrt{a})}
\]

\[
= \lim_{h \to 0} \frac{5}{\sqrt{a + h} + \sqrt{a}} = \frac{5}{2\sqrt{a}}.
\]

Using the rule for sums, constants, constant multiples and the rule for powers of \( x \), with \( \sqrt{x} = x^{1/2} \),

\[
\frac{dy}{dx} = \frac{5}{2}x^{-1/2} = \frac{5}{2\sqrt{x}}
\]
and if you substitute \( x = a \) you get the same answer as before.

**Problem 7** Assume that the demand for Aaaah! brand perfume is given by

\[
D(p) = 1500 - 4.5p - 0.02p^2
\]
where \( p \) is the unit price in dollars.

(a) Find the revenue \( R(p) \) at unit price \( p \).
(b) Find the marginal revenue \( R'(p) \) at \( p = 40 \).
(c) Use your answer in part (b) to approximate the additional revenue from raising the unit price of Aaaah! perfume from 40 to 43 dollars. **The exact answer will not give credit.**
(d) For what value(s) of \( p \) is the marginal revenue \( R'(p) \) zero? (the revenue is maximal there).

**Solution.**  
(a) \( R = pD(p) = 1500p - 4.5p^2 - 0.02p^3 \).
(b) First differentiate \( R(p) \) from part (a),

\[ R'(p) = 1500 - 9p - 0.06p^2. \]

Then substitute \( p = 40 \), so

\[ R'(40) = 1500 - 9 \cdot 40 - 0.06(40)^2 = 1500 - 360 - 96 = 1044. \]

(c) \( R(43) - R(40) \approx 3R'(40) = 3132 \), so about 3132 dollars additional revenue.
(d) Use the quadratic formula. But with paper and pencil, it’s helpful to first simplify!

\begin{align*}
1500 - 9p - 0.06p^2 &= 0 \quad \text{divide by } 3 \\
500 - 3p - 0.02p^2 &= 0 \quad \text{multiply by } 1/0.02 = 50 \\
25000 - 150p - p^2 &= 0 \\
p^2 + 150p - 25000 &= 0 \\
p &= \frac{-150 \pm \sqrt{150^2 + 10000}}{2} \quad \text{OK to stop here} \\
&= -75 \pm 5\sqrt{1225} = -75 \pm 175 \quad \text{gives } p = 100, -250 \\
(p + 250)(p - 100) &= 0 \quad \text{if you saw that, good for you!}
\end{align*}

So the only solution is \( p = 100 \) (negative answers have to be discarded).

**Problem 8**  
Assume that the total cost of manufacturing a number \( x \) of ‘WHAP’ flyswatters is

\[ C(x) = 250 + 0.02x. \]

(a) Find the average cost \( \bar{C}(x) \) of making a ‘WHAP’ flyswatter.
(b) What is the average cost \( \bar{C}(200) \) of making a ‘WHAP’ flyswatter if the total is \( x = 200 \)?
(c) Find the marginal average cost of making x 'WHAP' flyswatters.

Solution.  
(a) $\bar{C}(x) = \frac{250}{x} + 0.02$.
(b) $\bar{C}(200) = \frac{250}{200} + 0.02 = 1.27$.
(c) $\bar{C}'(x) = -\frac{250}{x^2}$.

Problem 9  
At unit price $p$, demand for one brand of mousetraps is given by

$$M(p) = 80 - 20p - 2p^2.$$  

(a) Find $M'(p)$ using a limit computation, NOT the Rules from Chapter 4. Then double-check using the Rules from Chapter 4.
(b) What is the instantaneous rate of change of $M$ at $p = 2.5$?

Solution.  
(a) Back to instantaneous rate of change $= \text{limit of average rate of change}$,

$$M'(p) = \lim_{h \to 0} \frac{M(p + h) - M(p)}{h}$$

$$= \lim_{h \to 0} \frac{80 - 20(p + h) - 2(p + h)^2 - (80 - 20p - 2p^2)}{h}$$

$$= \lim_{h \to 0} \frac{-20(p + h) - 2(p + h)^2 + 20p + 2p^2}{h}$$

$$= \lim_{h \to 0} \frac{-20h - 2(2ph + h^2)}{h}$$

$$= \lim_{h \to 0} -20 - 4p + 2h = -20 - 4p.$$  

Double-check using rules for constants, constant multiples, and powers, so

$$M'(p) = 0 - 20 \cdot 1 - 2 \cdot (2p) = -20 - 4p.$$  

(b) This is just about substituting $p = 2.5$ into the answer from part (a), so $M'(2.5) = -20 - 10 = -30$. 